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Study of Lateral Heat Loss Using Adomian Decomposition Method

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Abstract:

This paper presents the application of the decomposition method to solve fractional heat equations with lateral heat loss, subject to initial and boundary value conditions. Fractional heat equations, characterized by derivatives of non-integer order α, provide a powerful framework for modeling heat conduction with memory and hereditary effects. The inclusion of lateral heat loss adds complexity in the problem, making analytical solutions challenging. The Adomian Decomposition Method (ADM) is employed to construct the solution as an infinite series of rapidly converging terms.

Keywords: Adomian Decomposition Method, lateral heat loss, initial and boundary conditions.

Introduction:

Fractional Calculus (FC) is trending subject in researchers and scholars. It is the powerful subject by which real life problem can be modeled with precision. The fractional calculus is also develop at same time when the classical calculus developed, Fractional calculus is approximately three centuries old mathematical discipline. After the publication of the studies on Differential Calculus, where Leibnitz introduced the notation $\frac{d^n y}{dx^n}$; he received a letter from Bernoulli putting him a question about the meaning of a non-integer derivative order. Also he received a similar enquiry from L'Hôpital: What if n= $\frac{1}{2}$? Leibnitz's replay was prophetic: It will lead to a paradox, a paradox from which one day useful consequences will be drawn, because there are no useless paradoxes.[1-6] . After Leibniz many mathematician studied FC. It was Euler (1738) [3] who noticed the problem for a derivative of non-integer order. An integral representation for arbitrary order to define the derivative was suggested by Joseph Fourier (1822) [3, 5], and his version can be considered the first definition for the derivative of arbitrary (positive) order. Later on an integral equation associated with the tautochrone problem was solved by N.H.Abel (1826) [3, 5], which is the first application of FC. A definition based on the formula for differentiating the exponential function was suggested by Liouville (1832) [3, 5]. This expression is known as the first Liouville definition. Next definition formulated by Liouville is presented in terms of an integral and is now called the version by Liouville for the integration of non-integer order then the most important paper was published by Riemann [7]. We also note that both Liouville and Riemann formulations carry with them the so-called complementary function, a problem to be solved. Gr¨unwald [8] and

Letnikov [9], independently, developed an approach to non-integer order derivatives in terms of a convenient convergent series, conversely to the Riemann-Liouville approach, that is given by an integral. Letnikov showed that his definition coincides with the versions formulated by Liouville, for particular values of the order, and by Riemann, under a convenient interpretation of the so-called non-integer order difference. Hadamard (1892) [5] published a paper where the non-integer order derivative of an analytical function must be done in terms of its Taylor series.

Basic definitions of Fractional Calculus:

In literature there are many definitions of fractional derivatives [11-16] but the most frequently used are as given below

Definition:

Caputo fractional derivative with order α for a function f(t) is defined as

$$^{c}D^{\alpha}_{t0}(f(t))=\frac{1}{\Gamma(m-\alpha)}\int_{t0}^{t}(t-\tau)^{m-\alpha-1} f^{m}(\tau)\, d\tau$$

where $0 \le m - 1 \le \alpha < m$, $m \in Z_{+}$, and t = t0 is the initial time and $\Gamma(.)$ is the Gamma function.

Definition :

Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $x : (0, \infty) \rightarrow R$ is defined as

$$I^{\alpha}_{t0}(f(t)) = \frac{1}{\Gamma(\alpha)}\int_{t0}^{t}(t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

Where $t = t_0$ is the initial time and $\Gamma(.)$ is the Gamma function.

Definition :

Riemann-Liouville fractional derivative with order α for a function f(x): $(0, \infty) \rightarrow R$ is defined as

$$^{RL}D^{\alpha}_{t0}(f(t))=\frac{1}{\Gamma(m-\alpha)}\frac{d^{m}}{dx^{m}}\int_{t0}^{t}(t-\tau)^{m-\alpha-1} f(\tau)\, d\tau$$

Where $0 \le m - 1 \le \alpha < m$, $m \in Z_{+}$, and t = t0 is the initial time and $\Gamma(.)$ is the Gamma function.

Definition :

The Mittag-Leffler function is defined as

$$E_{\alpha}(Z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(ka+1)}$$

Where $\alpha > 0$, $z \in C$. The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(Z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(ka+\beta)},\quad \alpha,\beta > 0,\ z \in C$$

There are some properties between fractional-order derivatives and fractional order integrals, which are expressed as follows.

Properties:

Let $\alpha > 0$, $n = [\alpha]+1$ and $f_{n-\alpha}(t) = (I^{n-\alpha}{}_{a} f)(t)$

Then fractional integrals and fractional derivatives have the following properties.

(1) If $f(t) \in L^{1}(a; b)$ and $f_{n-\alpha}(t) \in AC^{n}[a; b]$, then

$$(I^{\alpha}_{a}\, {}^{RL}D^{\alpha}_{a}\, f)(t) = f(t) - \sum_{j=1}^{n}\frac{f^{(n-j)}{}_{n-\alpha}(a)}{\Gamma(a-j+1)}(t-a)^{\alpha-j},$$

holds almost everywhere in [a; b].

(2) If f(t∈ $AC^{n}[a; b]$, or $f(t) \in C^{n}[a; b]$, then

$(I^{\alpha}{}_a {}^cD^{\alpha}{}_a\ f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$.

Adomian Decomposition Method:

The Adomian decomposition method was introduced and developed by George Adomian in [16-17]. Which is helpful in solving linear and nonlinear differential equations, Huge research work has been done recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral equations as well.

The formal steps of the decomposition method to solve the one dimensional heat equation is as follows.

Without loss of generality, we study the initial-boundary value problem

$$\begin{aligned} &\text{PDE} \quad u_t = u_{xx} - u,\ 0 < x < \pi,\ t > 0, \\ &\text{BC} \quad u(0,t) = 0,\ t \geq 0, \\ &\qquad\ \ u(L,t) = 0,\ t \geq 0, \\ &\text{IC} \quad u(x,0) = f(x),\ 0 \leq x \leq \pi, \end{aligned} \qquad \text{(A)}$$

to achieve our goal.

To begin our analysis, we first rewrite (A) in an operator form by

$L_t\, u(x,t) = L_x\, u(x,t)$,

where the differential operators L_t and L_x are defined by

$$L_t = \partial/\partial t\ ,\ L_x = \partial^2/\partial x^2\ .$$

(B)

It is obvious that the integral operators $L^{-1}{}_t$ and $L^{-1}{}_x$ exist and may be regarded as one and two-fold definite integrals respectively defined by

$$L^{-1}{}_t\,(\cdot) = \int_0^t (\cdot)dt\ ,\ L^{-1}{}_x\,(\cdot) = \iint_{00}^{xx} (.)dxdx$$

(C)

This means that

$$L^{-1}{}_t\, L_t\, u(x,t) = u(x,t) - u(x,0).$$

(D)

Applying $L^{-1}{}_t$ to both sides of (B) and using the initial condition we find

$$u(x,t) = f(x) + L^{-1}{}_t\,(L_x\, u(x,t)).$$

(E)

The decomposition method defines the unknown function $u(x,t)$ into a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

(F)

One Dimensional Heat Flow:

where the components $u_0(x,t), u_1(x,t), u_2(x,t), \cdots$ are to be determined. Substituting (F) into both sides of (E) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L^{-1}{}_t\,\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) - \sum_{n=0}^{\infty} u_n(x,t)\right),$$

(G)

or equivalently

$$u_0+u_1+u_2+\cdots = f(x) + L^{-1}{}_t\,(L_x(u_0+u_1+u_2+\cdots) - (u_0+u_1+u_2+\cdots))$$

(H)

The decomposition method suggests that the zeroth component $u_0(x,t)$ is identified by the terms arising from the initial/boundary conditions and from source terms.

The remaining components of $u(x,t)$ are determined in a recursive manner such that each component is determined by using the previous component. Accordingly, we set the recurrence scheme

$$u_0(x,t) = f(x),$$

$$u_{k+1}(x,t) = L^{-1}_t\,(L_x\,(u_k(x,t)) - u_k(x,t))\ ,\ k \geq 0, \qquad \text{(I)}$$

for the complete determination of the components $un(x,t),n$ _ 0. In view of (G), the components $u_0(x,t),u_1(x,t),u_2(x,t),\cdots$ are determined individually by

$$u_0(x,t) = f(x),$$
$$u_1(x,t) = L^{-1}_t\,(L_x(u_0) - u_0\,),$$
$$u_2(x,t) = L^{-1}_t\,(L_x(u_1) - u_1\,)\,,$$
$$u_3(x,t) = L^{-1}_t\,(L_x(u_2) - u_2\,),$$

... (J)

Other components can be determined in a like manner as far as we like. The accuracy level can be effectively improved by increasing the number of components determined. Having determine -d the components $u_0,u_1,\cdots$, the solution $u(x,t)$ of the PDE is thus obtained in a series form given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n,$$

Obtained by substituting (I) into (D).[17,18]

Numerical Examples:

Example1:

P.D.E $D_t^{\propto} u\,(x,t) = u_{xx}(x,t) - u(x,y),$ **0 < x < π , t > 0** **(A)**

B.C. **u(0, t) = 0, t ≥ 0 ,**

u(1, t) = 0, t ≥ 0 ,

I.C. **u(x, 0) = sin(π x) + 2. sin (3 π x),**

Where $D_t^{\propto}$ denote Caputo derivative of order α.0 < α < 1.

Applying $D_t^{-\propto} = J_t^{\propto}$ on both side of the above equation (A).

$$J_t^{\propto}\,[D_t^{\propto} u] = J_t^{\propto}[\,u_{xx} - u\,]$$

$$u(x,t) = \sin(\pi x) + 2.\sin(3\pi x) + J_t^{\propto}[\,u_{xx} - u\,]$$

Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n$$

Hence,

$$\sum_{n=0}^{\infty} u_n = \sin(\pi x) + 2.\sin(3\pi x) + J_t^{\propto}[(\sum_{n=0}^{\infty} u_n)_{xx} - (\sum_{n=0}^{\infty} u_n)]$$

Recursive relation is defined by

$$u_0 = \sin(\pi x) + 2.\sin(3\pi x)$$
$$\sum_{k=0}^{\infty} u_{k+1} = J_t^{\propto}\,[(\sum_{n=0}^{\infty} u_k)_{xx} - (\sum_{n=0}^{\infty} u_k)]\ ,\ k \geq 0.$$
$$u_0 = \sin(\pi x) + 2.\sin(3\pi x)$$
$$u_1 = J_t^{\propto}\,[-(\pi^2+1)\sin(\pi x) - 2.(9\pi^2+1)\sin(3\pi x)\,]$$

$$u_1 = [-(\pi^2+1)\sin(\pi x) - 2.(9\pi^2+1)\sin(3\pi x)]\frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$u_2 = J_t^\alpha[\{(\pi^2+1)^2\sin(\pi x) + 2.(9\pi^2+1)^2\sin(3\pi x)\}\frac{t^\alpha}{\Gamma(\alpha+1)}]$$
$$u_2 = \{(\pi^2+1)^2\sin(\pi x) + 2.(9\pi^2+1)^2\sin(3\pi x)\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$u_3 = J_t^\alpha[\{-(\pi^2+1)^3\sin(\pi x) - 2.(9\pi^2+1)^3\sin(3\pi x)\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}]$$
$$u_3 = \{-(\pi^2+1)^3\sin(\pi x) - 2.(9\pi^2+1)^3\sin(3\pi x)\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$\vdots$$

$$u(x,y) = \sin(\pi x) + 2.\sin(3\pi x) - [(\pi^2+1)\sin(\pi x) + 2.(9\pi^2+1)\sin(3\pi x)]\frac{t^\alpha}{\Gamma(\alpha+1)} + \{(\pi^2+1)^2\sin(\pi x) + 2.(9\pi^2+1)^2\sin(3\pi x)\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \{(\pi^2+1)^3\sin(\pi x) + 2.(9\pi^2+1)^3\sin(3\pi x)\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \dots$$

$$u(x,t) = \sin(\pi x)[1 - (\pi^2+1)\frac{t^\alpha}{\Gamma(\alpha+1)} + (\pi^2+1)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (\pi^2+1)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots] + 2\sin(3\pi x)[1 - (9\pi^2+1)\frac{t^\alpha}{\Gamma(\alpha+1)} + (9\pi^2+1)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (9\pi^2+1)^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots]$$

which is very closed to the exact solution

$$u(x,t) = \sin(\pi x)\, e^{-(\pi^2+1)t} + 2\sin(3\pi x)\, e^{-(9\pi^2+1)t}$$

Example2:

P.D.E $D_t^\alpha u(x,t) = u_{xx}(x,t) - 3u(x,y) + 3,\ 0 < x < \pi,\qquad t > 0$ ***(B)***

B.C. $u(0,t) = 1,\quad t \ge 0,$

$u(1,t) = 1,\quad t \ge 0,$

I.C. $u(x,0) = 1 + \sin x,$

Where D_t^α denote Caputo derivative of order $\alpha . 0 < \alpha < 1$.

Applying $D_t^{-\alpha} = J_t^\alpha$ on both side of the above equation (B).

$J_t^\alpha\,[D_t^\alpha u] = J_t^\alpha[\,u_{xx} - 3u + 3\,]$

$u(x,t) = 1 + \sin x + J_t^\alpha[\,u_{xx} - 3u + 3\,]$

Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n$$

Hence,

$$\sum_{n=0}^{\infty} u_n = 1 + \sin x + J_t^\alpha[(\sum_{n=0}^{\infty} u_n)_{xx} - 3(\sum_{n=0}^{\infty} u_n) + 3]$$

Recursive relation is defined by

$$u_0 = 1 + \sin x$$
$$\sum_{k=0}^{\infty} u_{k+1} = J_t^\alpha[(\sum_{n=0}^{\infty} u_k)_{xx} - (\sum_{n=0}^{\infty} u_k)],\ k \ge 0.$$
$$u_0 = 1 + \sin x$$

$$u_1 = J_t^{\propto} [-4 \sin x]$$

$$u_1 = -4 \sin x \frac{t^{\propto}}{\Gamma(\propto+1)}$$

$$u_2 = J_t^{\propto} [(16 \sin x) \frac{t^{\propto}}{\Gamma(\propto+1)} + 3]$$

$$u_2 = 16 \sin x \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 3 \frac{t^{\propto}}{\Gamma(\propto+1)}$$

$$u_3 = J_t^{\propto} [-64 \sin x \frac{t^{2\propto}}{\Gamma(2\propto+1)} - 9 \frac{t^{\propto}}{\Gamma(\propto+1)} + 3]$$

$$u_3 = -64 \sin x \frac{t^{3\propto}}{\Gamma(3\propto+1)} - 9 \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 3 \frac{t^{\propto}}{\Gamma(\propto+1)}$$

$$\vdots$$

$$u(x,y) = 1 + \sin x - 4 \sin x \frac{t^{\propto}}{\Gamma(\propto+1)} + 16 \sin x \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 3 \frac{t^{\propto}}{\Gamma(\propto+1)} - 64 \sin x \frac{t^{3\propto}}{\Gamma(3\propto+1)} - 9 \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 3 \frac{t^{\propto}}{\Gamma(\propto+1)} \dots$$

$$u(x,y) = 1 + \sin x [1 - 4 \frac{t^{\propto}}{\Gamma(\propto+1)} + 16 \frac{t^{2\propto}}{\Gamma(2\propto+1)} - 64 \frac{t^{3\propto}}{\Gamma(3\propto+1)} + \cdots] + 6 \frac{t^{\propto}}{\Gamma(\propto+1)} - 9 \frac{t^{2\propto}}{\Gamma(2\propto+1)} \dots$$

which is very closed to the exact solution

$u(x,t) = 1 + \sin x . e^{-4t}$

Conclusion:

Adomian Decomposition Method is a powerful tool to illustrate fractional linear and nonlinear differential equation and partial differential equation which reduces complexity of the problem with memory and hereditary effects. The method efficiently incorporates the initial and boundary conditions while addressing the effects of lateral heat loss. Numerical examples illustrate the accuracy and reliability of the ADM in solving such equations.

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