An Efficient Approach to Solve the Fractional Assignment Problem

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ABSTRACT

The Fractional Assignment Problem (FrAP) is an extension of the classical Assignment Problem, where the objective function is expressed as a ratio of two linear sums such as minimizing the ratio of total time to total output, production per worker, or inventory per sale. These scenarios frequently arise in real-world applications, including transportation, logistics, scheduling, and network optimization. The goal is to minimize total cost or maximize total profit while adhering to capacity and demand constraints. This study presents the theoretical foundations, mathematical formulation, and a novel computational approach for effectively solving the Fractional Assignment Problem.

Introduction

The Fractional Assignment Problem (FrAP) has its roots in the classical Assignment Problem, which emerged in the mid-20th century from the broader fields of linear programming and combinatorial optimization. The classical Assignment Problem, formally introduced by Kuhn (1955) through the *Hungarian Method* [1], focused on assigning n tasks to n agents in a one-to-one manner to minimize total cost or maximize total profit. While the classical model was highly effective for discrete, indivisible tasks, it soon became evident that many real-world scenarios required a more flexible allocation framework. Situations such as fractional resource distribution, multi-product manufacturing, partial workload sharing, and transportation logistics could not be adequately modeled using strict one-to-one assignments. The mathematical formulation of FrAP developed further in the 1960s and 1970s, alongside the growth of Linear Fractional Programming (LFrP). Researchers such as Charnes and Cooper (1962) [2] introduced transformations to linearize fractional objectives, paving the way for efficient solution methods applicable to FrAP. Subsequently, Dinkelbach (1967) [3] proposed an iterative algorithm for solving nonlinear fractional programs, which has been effectively adapted for assignment-type problems.

A distinctive feature of the FrAP is its objective function, expressed as a ratio of two linear functions — for instance, minimizing total time to total output, production per worker, or inventory per sale. These ratio-based objectives naturally arise in performance optimization problems across diverse domains, including transportation, logistics, scheduling, production planning, and network design. The FrAP thus captures more realistic and complex relationships between costs, outputs, and efficiency measures than traditional assignment models.

Mathematically, the FrAP can be formulated as a linear fractional programming problem, where both the numerator and denominator are linear functions of the decision variables. Different approaches to solve linear FrAP were proposed by Odior, Pandey, and Punnen [11]. Sharma et al. [10] used an extension of the simplex technique to solve FrAP. Pandian and Jayalakshmi [4] developed an algorithm using a linear programming approach, while Doke [5], [6] introduced methods based on Taylor series expansion for multi-objective fractional and linear fractional transportation problems. Further advancements were made by Munot and Ghadle [7], [8], who applied the concept of modular arithmetic to solve the Assignment Problem(AP)[9], Transportation Problem (TP), later extending it to the Fractional Transportation Problem (FrTP).

In this article, the author presents a novel approach to solve the Fractional Assignment Problem using the Ghadle–Munot algorithm, which is based on modular arithmetic [7], [8]. This research focuses on the theoretical foundations, mathematical modeling, and development of a computational approach for solving the FrAP, supported by a numerical illustration. The study further explores its practical applications in real-world optimization scenarios, highlighting the problem's superiority in flexibility, applicability, and efficiency compared to the classical assignment model.

Preliminaries

Fractional Assignment Problem

The FrAP is a variant of the classical AP in which the objective function is expressed as a ratio (fraction) of two linear functions of the decision variables, rather than a single linear sum.

Mathematical Model of FrAP

An expression of FrAP is as follows:

$$\text{Max } Z(x) = \frac{P(x)}{T(x)} = \frac{Max\ Z_1(x)}{Min\ Z_2(x)} = \frac{\sum_{a=1}^{m}\sum_{b=1}^{m} p_{ab}x_{ab}}{\sum_{a=1}^{m}\sum_{b=1}^{m} t_{ab}x_{ab}};$$

Or

Min $Z(x) = \frac{P(x)}{T(x)} = \frac{Min\ Z_1(x)}{Max\ Z_2(x)} = \frac{\sum_{a=1}^{m}\sum_{b=1}^{m} p_{ab}x_{ab}}{\sum_{a=1}^{m}\sum_{b=1}^{m} t_{ab}x_{ab}}$

With,

$\sum_{a=1}^{m} x_{ab} = 1;\quad for\ a = 1,2,\dots,m$ ------ (i)

$\sum_{b=1}^{m} x_{ab} = 1;\quad for\ b = 1,2,\dots,m$ ------ (ii)

And $x_{ab} = 0\ or\ 1$, $\forall\ a = 1,2,\dots,m\ \&\ b = 1,2,\dots,m$ ------ (iii)

Where,

x_{ab} – Assignment of machine a to job b

p_{ab} - maximum cost of product given by machine

t_{ab} – minimum benefit gained from product by machine

Here, we want to get optimal solution of objective function Z(x), satisfying (i) to (iii).

Proposed Ghadle-Munot algorithm to solve Fractional Assignment Problem:

I. Let Z(x) be given FrAP. Consider two separate AP, *P(x)* and *B(x)* from given Maximization Z(x) [for Minimization Z(x)] where
P(x) = Max $Z_1(x) = \sum_{a=1}^{m}\sum_{b=1}^{n} p_{ab}x_{ab}$,
[for minimization *P(x)* = Min $Z_1(x) = \sum_{a=1}^{m}\sum_{b=1}^{n} p_{ab}x_{ab}$]
Subject to (i) to (iii).
And
B(x) = Min $Z_2(x) = \sum_{a=1}^{m}\sum_{b=1}^{n} t_{ab}x_{ab}$,
[for maximization *T(x)* = Max $Z_2(x) = \sum_{a=1}^{m}\sum_{b=1}^{n} t_{ab}x_{ab}$]
Subject to (i) to (iii).

II. Obtain feasible solution for *P(x)* using Ghadle-Munot algorithm to solve AP which uses modular arithmetic, Using the same allocation pattern, compute the corresponding allocation of benefit $T(x)$ and record the resulting value of $T(x)$.

Next, determine the ratio, $R_1 = \frac{P(x)}{T(x)}$.

III. Now, generate another feasible solution for $B(x)$ through the Ghadle–Munot modular arithmetic method. With these allocations, calculate $T(x)$ first, and afterward, evaluate the $P(x)$for the same allocations.
Calculate the second ratio as, $R_2 = \frac{P(x)}{T(x)}$.

IV. For maximization problems, if $R_1 > R_2$, proceed to Step VII; otherwise, go to Step VIII.

V. For minimization problems, if $R_1 < R_2$, go to Step VIII; otherwise, proceed to Step VII.

VI. If $R_1 = R_2$, the obtained solution is the optimal or best settlement solution.

VII. Solution corresponding to R_1 is best settlement solution.

VIII. Solution corresponding to R_2 is best settlement solution.

Now we will illustrate this algorithm by numerical examples.

Numerical Example

A beverages production dealer manufactures its products from 3 different machines with 3 different capacities. Its cost of production and benefit after production is given in table as upper right and lower left part of diagonal of each cell respectively. Solve the example to minimize the objective function.

Machine / Work	M_1	M_2	M_3	Availability
W_1	10 / 2	12 / 3	9 / 1	1
W_2	8 / 4	7 / 2	11 / 5	1
W_3	9 / 3	14 / 1	6 / 2	1
Requirement	1	1	1	

Solution:

Here we want to minimize the objective function therefore as per proposed algorithm to solve fractional assignment problem we proceed.

Step I.

$$\text{Min } Z(x) = \frac{P(x)}{T(x)} = \frac{Min\ Z_1(x)}{Max\ Z_2(x)} = \frac{\sum_{a=1}^{m}\sum_{b=1}^{m} p_{ab}x_{ab}}{\sum_{a=1}^{m}\sum_{b=1}^{m} t_{ab}x_{ab}}$$

Subject to constraint,

$\sum_{a=1}^{m} x_{ab} = 1; \quad for\ a = 1,2,\dots,m$ ------ (i)

$\sum_{b=1}^{m} x_{ab} = 1; \quad for\ b = 1,2,\dots,m$ ------ (ii)

And $x_{ab} = 0\ or\ 1,\ \forall\ a = 1,2,\dots,m\ \&\ b = 1,2,\dots,m$ ------ (iii)

Where,

x_{ab} – Assignment of machine a to job b

p_{ab} - maximum cost of product given by machine

t_{ab} – minimum benefit gained from product by machine

Step II.

First consider product cost matrix and solve it by Ghadle – Munot Algorithm for Minimization of Assignment Problem,

Machine / Work	M_1	M_2	M_3	Availability
W_1	10	12	9	1
W_2	8	7	11	1
W_3	9	14	6	1
Requirement	1	1	1	

By Ghadle-Munot algorithm we get IBFS as x_{11}=10;x_{22}=7; x_{33}=6; with minimum allocation *P(x)*= 23.
Using these same allocations for benefit matrix we get *B(x)*= 6.
Its ratio $R_1 = \frac{23}{6} = 3.833$
Step III.

Now consider benefit matrix and solve it by Ghadle-Munot algorithm for maximization of Assignment Problem.

Machine / Work	M_1	M_2	M_3	Availability
W_1	2	3	1	1
W_2	4	2	5	1
W_3	3	1	2	1
Requirement	1	1	1	

By Ghadle-Munot algorithm we get IBFS as x_{12}=3;x_{21}=4; x_{33}=2; with maximum allocation *B(x)*= 9.
Using these same allocations for product matrix we get *P(x)*= 26.
Its ratio $R_2 = \frac{26}{9} = 2.888$
Step IV.

Here, $R_1 > R_2 \Rightarrow$ allocations corresponding to R_2 gives best settlement solution.

Conclusion:

The present study provides a comprehensive exploration of the FrAP and introduces a novel solution approach utilizing the Ghadle–Munot Algorithm, which incorporates the principles of modular arithmetic. The study demonstrates through theoretical formulation and numerical illustration that the modular arithmetic-based Ghadle–Munot algorithm can successfully yield feasible and near-optimal solutions with reduced computational effort compared to conventional iterative methods like Charnes–Cooper and Dinkelbach algorithms. Moreover, the use of ratio analysis provides an insightful decision-making framework for evaluating competing allocations under different optimization criteria. This comparative approach enhances the algorithm's flexibility in both maximization and minimization contexts**,** offering a balanced settlement solution.

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