Cosmos Multidisciplinary Research E-Journal
Online Available at www.cmrj.in
Recognized International Peer Reviewed Journal
Impact Factor 4.94
ISSN No. 2456-1665

Adomian Decomposition Method: A Reliable Analytical Tool for Solving Multi-Dimensional Fractional Heat Equations

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Abstract:

In this paper, Application of the decomposition method to solve the fractional two-dimensional inhomogeneous heat equation in respect of initial and boundary value conditions has been studied. Fractional derivatives, which extend classical calculus to model memory and hereditary effects, provide a versatile framework for describing heat transfer in complex systems. The Adomian Decomposition Method (ADM) is employed to derive an analytical solution in the form of a convergent series, effectively handling the nonlinearity and inhomogeneity of the equation. Numerical examples illustrate the accuracy and efficiency of the ADM in capturing the dynamic behavior of the system under inhomogeneous conditions.

Keywords: *fractional two-dimensional inhomogeneous heat equation, Adomian Decomposition Method, initial and boundary value.*

Introduction:

The theory of Fractional Order Calculus (FOC) is almost three century old discipline, but researchers could able to use it in the last two decades on account of available of computational recourses. Fractional calculus was introduced on September 30, 1695. On that day, Leibniz wrote a letter to L'Hopital, raising the possibility of generalizing the meaning of derivatives from integer order to non-integer order derivatives. L'Hopital wanted to know the result for the derivative of order n = 1/2. Leibniz replied that "one day, useful consequences will be drawn" and in fact, his vision became a reality. Lacroix presented a definition of fractional derivative based on the usual expression for the n^{th} derivative of the power function (Lacroix 1819). In very short period the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional (i.e., non-integer) differential operators were introduced: the Grunwald-Letnikow, Riemann-Liouville, Hadamard, Caputo, Riesz (Hilfer2000; Kilbas 2006; Podlubny1999; Samko 1993) and the more recent notions of Cresson (2007), Katugampola (2011), Klimek (2005), Kilbas and Saigo (2004) or variable order fractional operators introduced by Samko and Ross (1993). Researchers in some fields of applied science and in engineering including signal processing, controls and many other fields such as biological science and neuroscience have used some aspects of fractional derivatives and integrals in their work [1-6]

Preliminaries and Notations:

Fractional Calculus:

In literature there are many definitions of fractional derivatives [7-12] but the most frequently used are as given below

Caputo fractional derivative with order α:

Caputo fractional derivative with order α for a function x(t) is defined as

$$^{c}D^{\alpha}_{t0}(x(t))=\frac{1}{\Gamma(m-\alpha)}\int_{t0}^{t}(t-\tau)^{m-\alpha-1}\,x^{m}(\tau)\,d\tau$$

where $0 \le m - 1 \le \alpha < m$, $m \in Z_{+}$, and $t = t0$ is the initial time and $\Gamma(.)$ is the Gamma function.

Riemann-Liouville fractional integral of order :

Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $x : R+ \to R$ is defined as

$$I^{\alpha}_{t0}\,(f(t)) = \frac{1}{\Gamma(\alpha)}\int_{t0}^{t}(t-\tau)^{\alpha-1}\,f(\tau)\,d\tau$$

Where $t = t_0$ is the initial time and $\Gamma(.)$ is the Gamma function.

Riemann-Liouville fractional derivative with order α :

Riemann-Liouville fractional derivative with order α for a function $x : R^{+} \to R$ is defined as

$$^{RL}D^{\alpha}_{t0}(x(t))=\frac{1}{\Gamma(m-\alpha)}\frac{d^{m}}{dx^{m}}\int_{t0}^{t}(t-\tau)^{m-\alpha-1}\,x\,(\tau)\,d\tau$$

Where $0 \le m - 1 \le \alpha < m$, $m \in Z_{+}$, and $t = t0$ is the initial time and $\Gamma(.)$ is the Gamma function.

Mittag-Leffler function :

The Mittag-Leffler function is defined as

$$E_{\alpha}(Z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+1)}$$

Where $\alpha > 0$, $z \in C$. The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(Z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+\beta)},\quad \alpha,\beta > 0,\ z \in C$$

There are some properties between fractional-order derivatives and fractional order integrals, which are expressed as follows.

Properties:

Let $\alpha > 0$, $n = [\alpha]+1$ and $f_{n-\alpha}(t) = (I^{n-\alpha}{}_{a} f)(t)$

Then fractional integrals and fractional derivatives have the following properties.

(1) If $f(t) \in L^{1}(a; b)$ and $f_{n-\alpha}(t) \in AC^{n}[a; b]$, then

$$(I^{\alpha}{}_{a}\,{}^{RL}D^{\alpha}{}_{a}\ f)(t) = f(t) - \sum_{j=1}^{n}\frac{f^{(n-j)}{}_{n-\alpha}(a)}{\Gamma(a-j+1)}(t-a)^{\alpha-j},$$

holds almost everywhere in [a; b].

(2) If $f(t \in AC^{n}[a; b]$, or $f(t) \in C^{n}[a; b]$, then

$$(I^{\alpha}{}_{a}\,{}^{c}D^{\alpha}{}_{a}\ f)(t) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^{k}.$$

Adomian Decomposition Method:

The method of decomposition is introduce by Adomian [13] which has been successfully used in solving a linear and nonlinear, ordinary, and partial differential equations [13-14]. Method has several distinct advantages. Firstly, approximate analytical solutions in the form of power series can be derived quickly, easily, and accurately, even for nonlinear equations. Another feature that distinguishes this method from those based on linearization or perturbation techniques.

The distribution of heat flow in a two dimensional space is governed by the following initial boundary value problem [25,26]

PDE $u_t = k(u_{xx} + u_{yy}) + p(x,y)$, $0 < x < a$, $0 < y < b$, $t > 0$,

B.C. $u(0, y, t) = u(a, y, t) = 0$,

$u(x, 0, t) = u(x, b, t) = 0$,

$$I.C.\ u(x,y,0) = f(x,y), \qquad \text{(A)}$$

where $u \equiv u(x,y,t)$ is the temperature of any point located at the position (x,y) of a rectangular plate at any time t, and k is the thermal diffusivity.

The solution in the t space, the x space, or the y space will produce the same series solution. However, the solution in the t space reduces the size of calculations compared with the other space solutions. For this reason the solution in the t direction will be followed in this chapter.

To begin our analysis, we first rewrite (A) in an operator form by

$$L_t\, u(x,y,t) = \mathrm{k}(\, L_x\, u(x,y,t) + L_y\, u(x,y,t)) + \mathrm{p(x,y)},$$

where the differential operators L_t , L_x and L_y are defined by

$$L_t = \partial/\partial\, t\ ,\ L_x = \partial^2/\partial x^2\ ,\ L_y = \partial^2/\partial y^2. \qquad \text{(B)}$$

It is obvious that the integral operators $L^{-1}{}_t$, $L^{-1}{}_x$ and $L^{-1}{}_y$ exist and may be regarded as one and two-fold definite integrals respectively defined by

$$L^{-1}{}_t\,(\cdot) = \int_0^t (\cdot)dt\ ,\ L^{-1}{}_x\,(\cdot) = \iint_{00}^{xx} (.)dxdx\ ,\ L^{-1}{}_y\,(\cdot) = \iint_{00}^{yy} (.)dydy \qquad \text{(C)}$$

This means that

$$L^{-1}{}_t\, L_t\, u(x,y,t) = u(x,y,t) - u(x,y,0). \qquad \text{(D)}$$

Applying $L^{-1}{}_t$ to both sides of (B) and using the initial condition we find

$$u(x,y,t) = f\,(x,y) + L^{-1}{}_t\,(\mathrm{p(x,y)}) + L^{-1}{}_t\,(\mathrm{k}(L_x\, u(x,y,t) + L_y\, u(x,y,t)). \qquad \text{(E)}$$

The decomposition method defines the unknown function $u(x,t)$ into a sum of components defined by the series

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(\mathrm{x,y,t}), \qquad \text{(F)}$$

Two Dimensional Heat Flow:

Where the components $u_0(x,t), u_1(x,t), u_2(x,t), \cdots$ are to be determined. Substituting (F) into both sides of (E) yields

$$\sum_{n=0}^{\infty} u_n(\mathrm{x,y,t}) = f\,(x) + L^{-1}{}_t\,(\mathrm{p(x,y)}) + L^{-1}{}_t\,(\mathrm{k}(L_x(\sum_{n=0}^{\infty} u_n(\mathrm{x,y,t})\ + L_y(\, \sum_{n=0}^{\infty} u_n(\mathrm{x,y,t})), \qquad \text{(G)}$$

or equivalently

$$u_0+u_1+u_2+\cdots = f\,(x)+L^{-1}{}_t\,(\mathrm{p(x,y)})+\mathrm{k}\,(L^{-1}{}_t\,(L_x(u_0+u_1+u_2+\cdots) - L^{-1}{}_t\,(L_y\,(u_0+u_1+u_2+\cdots)) \qquad \text{(H)}$$

The decomposition method suggests that the zeroth component $u_0(x,y,t)$ is identified by the terms arising from the initial/boundary conditions and from source terms.

The remaining components of $u(x,y,t)$ are determined in a recursive manner such that each component is determined by using the previous component. Accordingly, we set the recurrence scheme

$$u_0(x,y,t) = f\,(x,y) + L^{-1}{}_t\,(\mathrm{p(x,y)}),$$

$$u_{k+1}(x,y,t) = \mathrm{k}\ L^{-1}{}_t\,(L_x\,(u_k(x,y,t)) + L^{-1}{}_t\,(L_y\, u_k(x,y,t))\ ,\ k \geq 0, \qquad \text{(I)}$$

for the complete determination of the components $u_n(x,y,t), n \geq 0$. In view of (G), the components $u_0(x,y,t), u_1(x,y,t), u_2(x,y,t), \cdots$ are determined individually by

$$u_0(x,y,t) = f\,(x,y) + L^{-1}{}_t\,(\mathrm{p(x,y)}),$$
$$u_1(x,y,t) = L^{-1}{}_t\,(L_x(u_0) - u_0\,),$$
$$u_2(x,y,t) = L^{-1}{}_t\,(L_x(u_1) - u_1\,)\,,$$
$$u_3(x,y,t) = L^{-1}{}_t\,(L_x(u_2) - u_2\,),$$
$$\ldots \qquad \text{(J)}$$

Other components can be determined in a like manner as far as we like. The accuracy level can be effectively improved by increasing the number of components determined. Having determine -d the components $u_0, u_1, \cdot\cdot\cdot$, the solution $u(x,y,t)$ of the PDE is thus obtained in a series form given by

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n,$$

Obtained by substituting (I) into (D)

Numerical Examples:

Example 1:

P.D.E $D_t^{\propto} u\,(x,y,t) = 2\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right) + 2\sin x, \quad 0 < x, y < \pi, t > 0 \quad$ **(5.1)**

B.C. $u(0,y,t) = u(\pi,y,t) = 0,$

$u(x,0,t) = u(x,\pi,t) = \sin x,$

I.C. $u(x,y,0) = \sin x \sin y + \sin x$

Where $D_t^{\propto}$ denote Caputo derivative of order $\alpha . 0 < \alpha < 1$.

Applying $D_t^{-\propto} = J_t^{\propto}$ on both side of the above equation (5.1).

$$J_t^{\propto}\,[D_t^{\propto} u] = J_t^{\propto}[2(u_{xx} + u_{yy}) + 2\sin x\,]$$

$$u(x,y,t) = \sin x \sin y + \sin x + J_t^{\propto}[2(u_{xx} + u_{yy}) + 2\sin x\,]$$

Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n$$

Hence,

$$\sum_{n=0}^{\infty} u_n = \sin x \sin y + \sin x + J_t^{\propto}\left[2((\sum_{n=0}^{\infty} u_n)_{xx} + (\sum_{n=0}^{\infty} u_n)_{yy}\right) + 2\sin x]$$

Recursive relation is defined by

$$u_0(x,y,t) = \sin x \sin y + \sin x$$

$$\sum_{k=0}^{\infty} u_{k+1}(x,y,t) = J_t^{\propto}[\left[2((\sum_{k=0}^{\infty} u_k)_{xx} + (\sum_{k=0}^{\infty} u_k)_{yy}\right) + 2\sin x], k \geq 0.$$

$$u_0(x,y,t) = \sin x \sin y + \sin x$$

$$u_1(x,y,t) = J_t^{\propto}[-4\sin x \sin y\,]$$

$$u_1(x,y,t) = -4\sin x \sin y \frac{t^{\propto}}{\Gamma(\propto +1)}$$

$$u_2(x,y,t) = J_t^{\propto}[16\sin x \sin y \frac{t^{\propto}}{\Gamma(\propto +1)} + 2\sin x]$$

$$u_2(x,y,t) = 16\sin x \sin y \frac{t^{2\propto}}{\Gamma(2\propto +1)} + 2\sin x \frac{t^{\propto}}{\Gamma(\propto +1)}$$

$$u_3(x,y,t) = J_t^{\propto}[-64\sin x \sin y \frac{t^{2\propto}}{\Gamma(2\propto +1)} - 4\sin x \frac{t^{\propto}}{\Gamma(\propto +1)} + 2\sin x]$$

$$u_3(x,y,t) = -64\sin x \sin y \frac{t^{3\propto}}{\Gamma(3\propto +1)} - 4\sin x \frac{t^{2\propto}}{\Gamma(2\propto +1)} + 2\sin x \frac{t^{\propto}}{\Gamma(\propto +1)}$$

$$\vdots$$

$u(x,y,t) = \sin x \sin y + \sin x - 4\sin x \sin y \frac{t^{\propto}}{\Gamma(\propto+1)} + 16\sin x \sin y \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 2\sin x \frac{t^{\propto}}{\Gamma(\propto+1)} - 64\sin x \sin y \frac{t^{3\propto}}{\Gamma(3\propto+1)} - 4\sin x \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 2\sin x \frac{t^{\propto}}{\Gamma(\propto+1)} + \cdots$

$$u(x,y,t) = \sin x \sin y \left(1 - 4\frac{t^{\propto}}{\Gamma(\propto+1)} + 16\frac{t^{2\propto}}{\Gamma(2\propto+1)} - 64\frac{t^{3\propto}}{\Gamma(3\propto+1)} + \cdots\right) + \sin x \left(1 + 4\frac{t^{\propto}}{\Gamma(\propto+1)} - 4\frac{t^{2\propto}}{\Gamma(2\propto+1)} + \cdots\right)$$

Exact solution is $u(x,y,t) = e^{-4t}\sin x \sin y + \sin x$

Example 2:

P.D.E $D_t^{\propto} u(x,y,t) = 3\big(u_{xx}(x,y,t) + u_{yy}(x,y,t)\big) + 3\cos x, \quad 0 < x, y < \pi, t > 0$ **(5.2)**

B.C. $u(0,y,t) = -u(\pi,y,t) = 1,$

$u(x,0,t) = u(x,\pi,t) = \cos x,$

I.C. $u(x,y,0) = \sin x \sin y + \cos x$

Where $D_t^{\propto}$ denote Caputo derivative of order α.0 < α < 1.
Applying $D_t^{-\propto} = J_t^{\propto}$ on both side of the above equation (5.2).

$J_t^{\propto}[D_t^{\propto} u] = J_t^{\propto}[3(u_{xx} + u_{yy}) + 3\cos x]$

$u(x,y,t) = \sin x \sin y + \cos x + J_t^{\propto}[3(u_{xx} + u_{yy}) + 3\cos x]$

Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n$$

Hence,

$$\sum_{n=0}^{\infty} u_n = \sin x \sin y + \cos x + J_t^{\propto}\left[3\left((\sum_{n=0}^{\infty} u_n)_{xx} + (\sum_{n=0}^{\infty} u_n)_{yy}\right) + 3\cos x\right]$$

Recursive relation is defined by

$$u_0(x,y,t) = \sin x \sin y + \cos x$$

$$\sum_{k=0}^{\infty} u_{k+1}(x,y,t) = J_t^{\propto}\left[\left[2\left((\sum_{k=0}^{\infty} u_k)_{xx} + (\sum_{k=0}^{\infty} u_k)_{yy}\right) + 3\cos x\right], k \geq 0.\right.$$

$$u_0(x,y,t) = \sin x \sin y + \cos x$$

$$u_1(x,y,t) = J_t^{\propto}[-6\sin x \sin y]$$

$$u_1(x,y,t) = -6\sin x \sin y \frac{t^{\propto}}{\Gamma(\propto+1)}$$

$$u_2(x,y,t) = J_t^{\propto}\left[36\sin x \sin y \frac{t^{\propto}}{\Gamma(\propto+1)} 3\cos x\right]$$

$$u_2(x,y,t) = 36\sin x \sin y \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 3\cos x \frac{t^{\propto}}{\Gamma(\propto+1)}$$

$$u_3(x,y,t) = J_t^{\propto}[-216 \sin x \sin y \frac{t^{2\propto}}{\Gamma(2\propto+1)} - 9\cos x \frac{t^{\propto}}{\Gamma(\propto+1)} + 3\cos x]$$

$$u_3(x,y,t) = -216 \sin x \sin y \frac{t^{3\propto}}{\Gamma(3\propto+1)} - 9\cos x \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 3\cos x \frac{t^{\propto}}{\Gamma(\propto+1)}$$

⋮

$$u(x,y,t) = \sin x \sin y + \cos x - 6 \sin x \sin y \frac{t^{\propto}}{\Gamma(\propto+1)} + 36 \sin x \sin y \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 3\cos x \frac{t^{\propto}}{\Gamma(\propto+1)} - 216 \sin x \sin y \frac{t^{3\propto}}{\Gamma(3\propto+1)} - 9\cos x \frac{t^{2\propto}}{\Gamma(2\propto+1)} + 3\cos x \frac{t^{\propto}}{\Gamma(\propto+1)} + \cdots$$

$$=$$

$$u(x,y,t) = \sin x \sin y \left(1 - 6\frac{t^{\propto}}{\Gamma(\propto+1)} + 36\frac{t^{2\propto}}{\Gamma(2\propto+1)} - 216\frac{t^{3\propto}}{\Gamma(3\propto+1)} + \cdots\right) + \cos x \left(1 + 6\frac{t^{\propto}}{\Gamma(\propto+1)} - 9\frac{t^{2\propto}}{\Gamma(2\propto+1)} + \cdots\right)$$

Exact solution is $u(x,y,t) = e^{-6t} \sin x \sin y + \cos x$

Conclusion:

The decomposition method used to solve the fractional two-dimensional inhomogeneous heat equation with initial and boundary value conditions. The Adomian Decomposition Method (ADM) is employed to derive an analytical solution in the form of a convergent series. The method integrates the initial and boundary conditions seamlessly into the solution process, while avoiding the limitations of discretization or perturbation techniques. The results confirm the decomposition method's capability as a reliable analytical tool for solving multi-dimensional fractional heat equations in real-world applications.

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