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Study of Fractional Two-Dimensional Heat Flow Equations with Initial and Boundary Value Conditions

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Abstract:

This paper explores the application of the decomposition method to solve fractional two-dimensional heat flow equations with initial and boundary value conditions. Fractional derivatives, which account for anomalous diffusion and memory effects, provide a robust mathematical framework for modeling heat transfer processes in two spatial dimensions. Numerical investigation has been done to study shows the efficacy and utility of ADM. Article explores the potential of the method for solving complex fractional differential equation ensuring accuracy and consistency.

Introduction:

In last few decades there is increasing interest in Fractional Calculus which deals with integration / differentiation of arbitrary orders. Almost all the field of science has applications of fractional calculus which has been ever growing and includes control theory, viscoelasticity, diffusion, turbulence, electromagnetism and many other physical processes. An exhaustive treatment of fractional calculus in this respect can be found in [1,2]. Several techniques such as Adomian decomposition method (ADM) [3], Homotopy perturbation method (HPM) [4], and a Variational iterative method have been developed for solving solving fractional differential equations in particular.

History of fractional calculus is available in [5-10]. Many researcher worked on FC in sequence After Leibniz, it was Euler (1738) [7] that noticed the problem for a derivative of non-integer order. Joseph Fourier (1822) suggested an integral representation for arbitrary order to define the derivative, and his version can be considered the first definition for the derivative of arbitrary (positive) order. Later on N.H.Abel (1826) solved an integral equation associated with the tautochrone problem, which is the first application of FC. Liouville (1832) [11, 13] suggested a definition based on the formula for differentiating the exponential function. This expression is known as the first Liouville definition. Next definition formulated by Liouville is presented in terms of an integral and is now called the version by Liouville for the integration of non-integer order then the most important paper was published by Riemann [11]. We also note that both Liouville and Riemann formulations carry with them the so-called complementary function, a problem to be solved. Gr¨unwald [12] and Letnikov [13], independently, developed an approach to non-integer order derivatives in terms of a convenient convergent series, conversely to the Riemann-Liouville approach, that is given by an integral. Letnikov showed that his definition coincides with the versions formulated by Liouville, for particular values of the order, and by Riemann, under a convenient interpretation of the so-called non-integer order difference. Hadamard (1892) [9] published a paper where the non-integer order derivative of an analytical function must be done in terms of its Taylor series.

Basic definitions Fractional Calculus:

To study basics 0f definitions of fractional derivatives and theory we refer reader [14-19] but the most frequently used are as given below

Caputo fractional derivative:

Caputo fractional derivative with order α, where $0 < \alpha \leq 1$ for a function f(t) is defined as

$$^{c}D^{\alpha}_{t0}(f(t))=\frac{1}{\Gamma(m-\alpha)}\int_{t0}^{t}(t-\tau)^{m-\alpha-1}\, f^{m}(\tau)\, d\tau$$

where $0 \leq m - 1 \leq \alpha < m$, $m \in Z_{+}$, and t = t0 is the initial time and $\Gamma(.)$ is the Gamma function.

Riemann-Liouville fractional integral:

Riemann-Liouville fractional integral of order $\alpha > 0$, $0 < \alpha \leq 1$for a function f(x) : $(0,\infty)\rightarrow R$ is defined as

$$I^{\alpha}_{t0}\,(f(t)) = \frac{1}{\Gamma(\alpha)}\int_{t0}^{t}(t-\tau)^{\alpha-1}\, f(\tau)\, d\tau$$

Where $t = t_0$ is the initial time and $\Gamma(.)$ is the Gamma function.

Riemann-Liouville fractional derivative:

Riemann-Liouville fractional derivative with order $\alpha, 0 < \alpha \leq 1$ for a function f(x) : $(0,\infty) \rightarrow R$ is defined as

$$^{RL}D^{\alpha}_{t0}(f(t))=\frac{1}{\Gamma(m-\alpha)}\frac{d^{m}}{dx^{m}}\int_{t0}^{t}(t-\tau)^{m-\alpha-1}\, f(\tau)\, d\tau$$

Where $0 \leq m - 1 \leq \alpha < m$, $m \in Z_{+}$, and t = t0 is the initial time and $\Gamma(.)$ is the Gamma function.

Mittag-Leffler function:

The Mittag-Leffler function is defined as

$$E_{\alpha}(Z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+1)}$$

Where $\alpha > 0$, $z \in C$. The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(Z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+\beta)},\quad \alpha,\beta > 0,\ z \in C$$

There are some properties between fractional-order derivatives and fractional order integrals, which are expressed as follows.

Properties:

Let $\alpha > 0$, $n = [\alpha]+1$ and $f_{n-\alpha}(t) = (I^{n-\alpha}{}_{a} f)(t)$

Then fractional integrals and fractional derivatives have the following properties.

(1) If $f(t) \in L^{1}(a; b)$ and $f_{n-\alpha}(t) \in AC^{n}[a; b]$, then

$$(I^{\alpha}_{a}\, {}^{RL}D^{\alpha}_{a}\, f)(t) = f(t) - \sum_{j=1}^{n}\frac{f^{(n-j)}{}_{n-\alpha}(a)}{\Gamma(a-j+1)}(t-a)^{\alpha-j},$$

holds almost everywhere in [a; b].

(2) If $f(t \in AC^{n}[a; b]$, or $f(t) \in C^{n}[a; b]$, then

$$(I^{\alpha}_{a}\, {}^{c}D^{\alpha}_{a}\, f)(t) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^{k}.$$

Adomian Decomposition Method:

The method of decomposition is introduce by Adomian [20] which has been successfully used in solving a linear and nonlinear, ordinary, and partial differential equations [21-24]. Method has several distinct advantages. Firstly, approximate analytical solutions in the form of power series can be derived quickly, easily, and accurately, even for nonlinear equations. Another feature that distinguishes this method from those based on linearization or perturbation techniques.

The distribution of heat flow in a two dimensional space is governed by the following initial boundary value problem [25,26]

$$\text{PDE } u_t = k(u_{xx} + u_{yy}),\ 0 < x < a,\ 0 < y < b,\ t > 0,$$
$$\text{B.C. } u(0,y,t) = u(a,y,t) = 0,$$
$$u(x,0,t) = u(x,b,t) = 0,$$
$$\text{I.C. } u(x,y,0) = f(x,y), \qquad \text{(A)}$$

where $u \equiv u(x,y,t)$ is the temperature of any point located at the position (x,y) of a rectangular plate at any time t, and k is the thermal diffusivity.
The solution in the t space, the x space, or the y space will produce the same series solution. However, the solution in the t space reduces the size of calculations compared with the other space solutions. For this reason the solution in the t direction will be followed in this chapter.
To begin our analysis, we first rewrite (A) in an operator form by

$$L_t\, u(x,y,t) = k(L_x\, u(x,y,t) + L_y\, u(x,y,t)),$$

where the differential operators L_t , L_x and L_y are defined by

$$L_t = \partial/\partial t,\ L_x = \partial^2/\partial x^2,\ L_y = \partial^2/\partial y^2. \qquad \text{(B)}$$

It is obvious that the integral operators $L^{-1}{}_t$, $L^{-1}{}_x$ and $L^{-1}{}_y$ exist and may be regarded as one and two-fold definite integrals respectively defined by

$$L^{-1}{}_t(\cdot) = \int_0^t (\cdot)dt,\ L^{-1}{}_x(\cdot) = \iint_{00}^{xx} (.)dxdx,\ L^{-1}{}_y(\cdot) = \iint_{00}^{yy} (.)dydy \qquad \text{(C)}$$

This means that

$$L^{-1}{}_t\, L_t\, u(x,y,t) = u(x,y,t) - u(x,y,0). \qquad \text{(D)}$$

Applying $L^{-1}{}_t$ to both sides of (B) and using the initial condition we find

$$u(x,y,t) = f(x,y) + L^{-1}{}_t\,(k(L_x\, u(x,y,t) + L_y\, u(x,y,t))). \qquad \text{(E)}$$

The decomposition method defines the unknown function $u(x,t)$ into a sum of components defined by the series

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t), \qquad \text{(F)}$$

Two Dimensional Heat Flow:

where the components $u_0(x,t), u_1(x,t), u_2(x,t), \cdots$ are to be determined. Substituting (F) into both sides of (E) yields

$$\sum_{n=0}^{\infty} u_n(x,y,t) = f(x) + L^{-1}{}_t\,(k(L_x(\sum_{n=0}^{\infty} u_n(x,y,t)\ + L_y(\sum_{n=0}^{\infty} u_n(x,y,t)), \qquad \text{(G)}$$

or equivalently

$$u_0 + u_1 + u_2 + \cdots = f(x) + k\,(L^{-1}{}_t\,(L_x(u_0 + u_1 + u_2 + \cdots) - L^{-1}{}_t\,(L_y\,(u_0 + u_1 + u_2 + \cdots)) \qquad \text{(H)}$$

The decomposition method suggests that the zeroth component $u_0(x,y,t)$ is identified by the terms arising from the initial/boundary conditions and from source terms.
The remaining components of $u(x,y,t)$ are determined in a recursive manner such that each component is determined by using the previous component. Accordingly, we set the recurrence scheme

$$u_0(x,y,t) = f(x,y),$$

$u_{k+1}(x,y,t) = \text{k } L^{-1}{}_t (L_x (u_k(x,y,t)) + L^{-1}{}_t (L_y u_k(x,y,t))$, $k \geq 0$, (I)

for the complete determination of the components $u_n(x,y,t), n \geq 0$. In view of (G), the components $u_0(x,y,t), u_1(x,y,t), u_2(x,y,t), \cdots$ are determined individually by

$$u_0(x,y,t) = f(x,y),$$
$$u_1(x,y,t) = L^{-1}{}_t (L_x(u_0) - u_0),$$
$$u_2(x,y,t) = L^{-1}{}_t (L_x(u_1) - u_1),$$
$$u_3(x,y,t) = L^{-1}{}_t (L_x(u_2) - u_2),$$
$$\ldots \qquad \text{(J)}$$

Other components can be determined in a like manner as far as we like. The accuracy level can be effectively improved by increasing the number of components determined. Having determine -d the components $u_0, u_1, \cdots$, the solution $u(x,y,t)$ of the PDE is thus obtained in a series form given by

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n,$$

Obtained by substituting (I) into (D)

Numerical Examples :

P.D.E $D_t^{\propto} u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t)$, $0 < x, y < \pi$, $t > 0$ **(A)**

B.C. $u(0,y,t) = u(\pi,y,t) = 0$,

$u(x,0,t) = u(x,\pi,t) = 0$,

I.C. $u(x,y,0) = \sin x \sin y$

Where $D_t^{\propto}$ denote Caputo derivative of order $\alpha . 0 < \alpha < 1$.

Applying $D_t^{-\propto} = J_t^{\propto}$ on both side of the above equation (A).

$J_t^{\propto} [D_t^{\propto} u] = J_t^{\propto} [u_{xx} + u_{yy}]$

$u(x,y,t) = \sin x \sin y + J_t^{\propto} [u_{xx} + u_{yy}]$

Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n$$

Hence,

$$\sum_{n=0}^{\infty} u_n = \sin x \sin y + J_t^{\propto} [(\sum_{n=0}^{\infty} u_n)_{xx} + (\sum_{n=0}^{\infty} u_n)_{yy}]$$

Recursive relation is defined by

$$u_0(x,y,t) = \sin x \sin y$$

$$\sum_{k=0}^{\infty} u_{k+1}(x,y,t) = J_t^{\propto} [(\sum_{k=0}^{\infty} u_k)_{xx} + (\sum_{k=0}^{\infty} u_k)_{yy}], \; k \geq 0.$$

$$u_0(x,y,t) = \sin x \sin y$$

$$u_1(x,y,t) = J_t^{\propto} [-2 \sin x \sin y]$$

$$u_1(x,y,t) = -2 \sin x \sin y \, \frac{t^{\propto}}{\Gamma(\propto +1)}$$

$$u_2(x,y,t) = J_t^{\propto}[4\sin x \sin y \frac{t^{\propto}}{\Gamma(\propto+1)}]$$

$$u_2(x,y,t) = 4\sin x \sin y \frac{t^{2\propto}}{\Gamma(2\propto+1)}$$

$$u_3(x,y,t) = J_t^{\propto}[-8\sin x \sin y \frac{t^{2\propto}}{\Gamma(2\propto+1)}]$$

$$u_3(x,y,t) = -8\sin x \sin y \frac{t^{3\propto}}{\Gamma(3\propto+1)}$$

$$\vdots$$

$$u(x,y,t) = \sin x \sin y\{1 + (-2)^1 \frac{t^{\propto}}{\Gamma(\propto+1)} + (-2)^2 \frac{t^{2\propto}}{\Gamma(2\propto+1)} + (-2)^3 \frac{t^{3\propto}}{\Gamma(3\propto+1)} + \cdots) \ldots$$

Exact solution is $u(x,y,t) = \sin x \sin y\, e^{-2t}$

P.D.E $\boldsymbol{D_t^{\propto} u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) - u(x,y,t)}$, $\boldsymbol{0 < x,y < \pi, t > 0}$ (B)

B.C. $\boldsymbol{u(0,y,t) = u(\pi,y,t) = 0}$,

$\boldsymbol{u(x,0,t) = -u(x,\pi,t) = e^{-3t}\sin x}$,

I.C. $\boldsymbol{u(x,y,0) = \sin x \cos y}$

Where $D_t^{\propto}$ denote Caputo derivative of order α.0 < α < 1.

Applying $D_t^{-\propto} = J_t^{\propto}$ on both side of the above equation (B).

$J_t^{\propto}[D_t^{\propto}u] = J_t^{\propto}[u_{xx} + u_{yy} - u]$

$u(x,y,t) = \sin x \cos y + J_t^{\propto}[u_{xx} + u_{yy} - u]$

Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n$$

Hence,

$$\sum_{n=0}^{\infty} u_n = \sin x \cos y + J_t^{\propto}[(\sum_{n=0}^{\infty} u_n)_{xx} + (\sum_{n=0}^{\infty} u_n)_{yy} - (\sum_{n=0}^{\infty} u_n)]$$

Recursive relation is defined by

$$u_0(x,y,t) = \sin x \cos y$$

$$\sum_{k=0}^{\infty} u_{k+1}(x,y,t) = J_t^{\propto}[(\sum_{k=0}^{\infty} u_k)_{xx} + (\sum_{k=0}^{\infty} u_k)_{yy} - (\sum_{n=0}^{\infty} u_k)],\ k \geq 0.$$

$$u_0(x,y,t) = \sin x \cos y$$

$$u_1(x,y,t) = J_t^{\propto}[-3\sin x \cos y]$$

$$u_1(x,y,t) = -3\sin x \cos y \frac{t^{\propto}}{\Gamma(\propto+1)}$$

$$u_2(x,y,t) = J_t^{\propto}[9\sin x \cos y \frac{t^{\propto}}{\Gamma(\propto+1)}]$$

$$u_2(x,y,t) = 9\sin x \cos y \frac{t^{2\propto}}{\Gamma(2\propto+1)}$$

$$u_3(x,y,t) = J_t^{\propto}[-27 \sin x \cos y \frac{t^{2\propto}}{\Gamma(2\propto+1)}]$$

$$u_3(x,y,t) = -27 \sin x \cos y \frac{t^{3\propto}}{\Gamma(3\propto+1)}$$

$$\vdots$$

$$u(x,y,t) = \sin x \cos y \{1 + (-3)^1 \frac{t^{\propto}}{\Gamma(\propto+1)} + (-3)^2 \frac{t^{2\propto}}{\Gamma(2\propto+1)} + (-3)^3 \frac{t^{3\propto}}{\Gamma(3\propto+1)} + \cdots) \ldots$$

Exact solution is $u(x,y,t) = e^{-3t}. \sin x \cos y$

Conclusion:

The Adomian Decomposition Method (ADM) is utilized to construct an analytical solution as a series of rapidly converging terms. This approach avoids the complexities associated with numerical methods, such as discretization and linearization, while preserving the intrinsic properties of the problem. The initial and boundary conditions are seamlessly incorporated into the iterative solution process, ensuring accuracy and consistency. Numerical examples are provided to demonstrate the ADM's efficacy in addressing two-dimensional fractional heat flow problems, highlighting its utility in advanced heat conduction modeling. The results emphasize the method's potential as a powerful analytical tool for solving complex fractional differential equations in multi-dimensional contexts.

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