A Study on Fixed Point Theorems of Generalized Cyclic Multi-Valued Mappings On Dislocated Quasi B-Metric Spaces

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1. Introduction and Preliminaries

In 2015, Chakkrid and Cholatis[1] introduced concept of dislocated quasi b-metric space and proved some fixed point theorems for cyclic contractions in such spaces. Mujeeb Ur Rahman and Muhammad Sarwar[3] studied dislocated quasi b-metric spaces and established fixed point theorems for Kannan contraction and Chetterjea type contraction. Cholatis et al.[2] proved fixed point theorems for cyclic weakly contractions in dislocated quasi b-metric space. They have also discussed topology of dislocated quasi b-metric spaces and studied some properties of dislocated quasi b-metric spaces. In 2017, Ampon Buayai, Araya Wiwatwanich, Annop Kaewkhao[4] established some fixed point theorems for cyclic multivalued mappings satisfying different contractive conditions in complete dislocated quasi metric spaces. In this paper we have established some fixed point theorems for different set valued contraction mappings in complete dislocated quasi b-metric spaces which extend some well known fixed point theorems existing in the literature.

Definition 1.1 *[1] Let X be a non-empty set. Let the mapping $d: X \times X \to [0, \infty)$ and constant $k \geq 1$ satisfy following conditions:*

1. $d(x, y) = 0 = d(y, x) \Rightarrow x = y, \forall x, y \in X$
2. $d(x, y) \leq k[d(x, z) + d(z, y)], \forall x, y, z \in X.$

Then the pair (X, d) is called dislocated quasi-b −metric space or in short dqb −metric space.

The constant k is called coefficient of (X, d). It is clear that b −metric spaces, quasi-b −metric spaces and b −metric-like spaces are dqb −metric spaces but converse is not true.

Example 1.2 *[3] Let $X = R^+$ and for $p > 1, d: X \times X \to [0, \infty)$ be defined as,*

$$d(x, y) = |x - y|^p + |x|^p, \forall x, y \in X.$$

Then (X, d) is dqb −metric space with $k = 2^p > 1$. But (X, d) is not b −metric space and also not dislocated quasi metric space.

Example 1.3 *[1] Let $X = R$ and suppose,*

$$d(x, y) = |2x - y|^2 + |2x + y|^2$$

then (X, d) is dqb −metric space with coefficient $k = 2$ but (X, d) is not a quasi-b −metric space. Also (X, d) is not dislocated quasi metric space.

Definition 1.4 *[4] A sequence $\{x_n\}$, dqb −converges to $x \in X$ if and only if for all $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for all $n > n_\epsilon, d(x, x_n) < \epsilon$. In this case, we write*

$$\lim_{n \to \infty} d(x, x_n) = 0.$$

In this case x is called dqb −limit of $\{x_n\}$ and $\{x_n\}$ is said to be dqb −convergent to x, written as $x_n \to x$.

Definition 1.5 *[4] A sequence $\{x_n\}$ is called a dqb −Cauchy sequence if and only if for all $\epsilon > 0$ there exist $n_\epsilon \in \mathbb{N}$ such that for all for all $m > n \geq n_\epsilon, d(x_n, x_m) < \epsilon$. In this case, we write $lim_{n\to\infty} d(x_n, x_m) = 0$.*

Definition 1.6 *[1] A dqb −metric space (X, d) is said to be dqb −complete if every dqb −Cauchy sequence in it is dqb −convergent in X.*

Definition 1.7 *[4] A subset M of X is said to be closed if and only if for each sequence $\{x_n\}$ in X which converges to an element x, we have $x \in M$.*

Definition 1.8 *[4]A subset S of X is bounded if there exists $M \in (0, \infty)$ such that $d(x, y) \leq M$ for all $x, y \in S$.*

Proposition 1.9 *[2] Every subsequence of a dqb −convergent sequence in a dqb −metric space (X, d) is dqb −convergent sequence.*

Proposition 1.10 *[2] Every subsequence of a dqb −Cauchy sequence in a dqb −metric space (X, d) is dqb −Cauchy sequence.*

Proposition 1.11 *[2]If (X, d) is a dqb −metric space then a function $f: X \to X$ is continuous if and only if $x_n \to x \Rightarrow fx_n \to fx$.*

Lemma 1.12 *[2] Limit of a dqb −convergent sequence in dqb −metric space is unique.*

Proposition 1.13 *If u is limit of some dqb −convergent sequence in a dqb −metric space (X, d) then $d(u, u) = 0$.*

Proof. $0 = \lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(x_n, x_m) = d(u, u), m > n$.

Ampon Buayai et.al[4] has defined following notions. Let (X, d) be a dqb −metric space with coefficient k. Let $A, B \in CB(X)$ where $CB(X)$ denote the family of all non-empty closed bounded subsets of X. Define the functional $H: CB(X) \times CB(X) \to R^+$ by

$$H(A, B) = \max\{h(A, B), h(B, A)\}$$

where

$$h(A, B) = \sup\{d(a, B): a \in A\}$$

with

$$d(a, B) = \inf\{d(a, b): b \in B\}.$$

Lemma 1.14 *[4] Let (X, d) be a dqb −metric space with coefficient k. For any $A, B \in CB(X)$ and any $x, y \in X$, the following statements are true:*

1. $d(x, B) \leq d(x, b)$ for any $b \in B$,
2. $h(A, B) \leq H(A, B)$,
3. $d(a, B) \leq H(A, B)$ for any $a \in A$,
4. $H(A, A) = 0$,
5. $H(A, B) = H(B, A)$,
6. $d(x, A) \leq k[d(x, y) + d(y, A)]$.

Lemma 1.15 *[4] Let (X,d) be a dqb –metric space with coefficient k and A be a non-empty closed subset of X and $x \in X$. If $d(x,A) = 0$, then $x \in A$.*

Lemma 1.16 *[4] Let (X,d) be a dqb –metric space with coefficient k and $A, B \in CB(X)$. If $H(A,B) = 0$, then $A = B$.*

Lemma 1.17 *[4] Let (X,d) be a dqb –metric space with coefficient k and $A, B \in CB(X)$. Then for each $c > 1$ and for all $x \in A$ there exists $y \in B$ such that $d(x,y) \le cH(A,B)$.*

Definition 1.18 *[6] Let (X,d) be a dqb –metric space and $T, S: X \to X$ be two mappings, then the mapping S is called T –contraction if there exists $\alpha \in [0,1)$ such that,*

$$d(TSx, TSy) \le \alpha d(Tx, Ty), \forall x, y \in X.$$

We define following contractions.

Definition 1.19 *Let (X,d) be a dqb –metric space. Let $T: X \to X$ and $S: A \cup B \to CB(X)$, where $A, B \in CB(X)$, be two mappings, then the mapping S is called T –Banach cyclic multivalued contraction if $S(A) \subset B, S(B) \subset A$ and there exists $\alpha \in [0,1)$ such that,*

$$H(TSx, TSy) \le \alpha d(Tx, Ty), \forall x, y \in X.$$

Definition 1.20 *Let (X,d) be a dqb –metric space. Let $T: X \to X$ and $S: A \cup B \to CB(X)$, where $A, B \in CB(X)$, be two mappings, then the mapping S is called T –Kannan cyclic multivalued contraction if $S(A) \subset B, S(B) \subset A$ and there exists $\alpha \in [0,1/2)$ such that,*

$$H(TSx, TSy) \le \alpha[d(Tx, TSx) + d(Ty, TSy)], \forall x, y \in X.$$

2. Main Results

Ampon Buayai, Araya Wiwatwanich, Annop Kaewkhao[4] have proved following two theorems for cyclic multi-valued Banach contraction mappings and cyclic multi-valued Kannan mappings in dqb –metric spaces.

Theorem 2.1 *[4] Let A and B be non-empty closed subsets of a complete dqb –metric space (X,d) with coefficient k and $T: A \cup B \to CB(X)$ be a cyclic multi-valued mapping. If there exists a constant $\alpha \in (0,1/2)$ with $k\alpha < 1/2$ such that*

$$H(Tx, Ty) \le \alpha[d(x, Tx) + d(y, Ty)],$$

for all $x \in A$ and $y \in B$. Then T has atleast one fixed point in $A \cap B$.

Theorem 2.2 *[4] Let A and B be non-empty closed subsets of a complete dqb –metric space (X,d) with coefficient k and $T: A \cup B \to CB(X)$ be a cyclic multi-valued mapping. If there exists a constant $\alpha \in (0,1)$ with $k\alpha < 1$ such that*

$$H(Tx, Ty) \le \alpha d(x, y),$$

for all $x \in A$ and $y \in B$. Then T has atleast one fixed point in $A \cap B$.

Now we state and prove our first result.

Theorem 2.3 *Let A and B be non-empty closed subsets of a complete dqb –metric space (X,d) with coefficient k and $f: A \cup B \to CB(X)$ be a cyclic multi-valued mapping. If there exists a function $\alpha: [0,\infty) \to (0,1/k)$ such that $H(fx, fy) \le \alpha(d(x,y))d(x,y)$, for all $x \in A$ and $y \in B$. Then f has atleast one fixed point in $A \cap B$.*

Proof. Let $x_0 \in X$ be a arbitrarily fixed. Let c be such that $1 < c < \frac{1}{sup\{\alpha(d(x,y)):x\in A, y\in B\}}$. We can choose $x_1 \in fx_0 \subset B$. By lemma (1.17), there exists $x_2 \in fx_1 \subset A$ such that

$$d(x_1,x_2) \le cH(fx_0,fx_1)$$
$$\le c\alpha(d(x_0,x_1))d(x_0,x_1).$$

Again using same argument as above, there exists $x_3 \in fx_2 \subset B$ such that

$$d(x_2,x_3) \le cH(fx_1,fx_2)$$
$$\le c\alpha(d(x_1,x_2))d(x_1,x_2)$$
$$\le cd(x_1,x_2)$$
$$\le c^2\alpha(d(x_0,x_1))d(x_0,x_1).$$

For simplicity let $\alpha^n = \alpha^n(d(x_0,x_1))$. Thus for any $n \in N$, we have $x_{n+1} \in fx_n$ such that

$$d(x_n,x_{n+1}) \le c^n\alpha^n d(x_0,x_1)$$
$$= s^n d(x_0,x_1), where \quad s = c\alpha < 1.$$

Now we prove that $\{x_n\}$ is a dqb –Cauchy sequence. Let $m,n \in N$ such that $m > n$ and $m = n + j$ for some $j \in N$. Consider,

$$d(x_n,x_m) = d(x_n,x_{n+j})$$
$$\le kd(x_n,x_{n+1}) + k^2d(x_{n+1},x_{n+2}) + \cdots + k^jd(x_{n+j-1},x_{n+j})$$
$$\le ks^nd(x_0,x_1) + k^2s^{n+1}d(x_0,x_1) + \cdots + k^js^{n+j-1}d(x_0,x_1)$$
$$\le [1 + ks + (ks)^2 + \cdots + (ks)^{j-1}]ks^nd(x_0,x_1)$$
$$= \frac{1-(ks)^j}{1-ks}ks^nd(x_0,x_1).$$

Taking limit as $n \to \infty$ in above inequality, we get

$$\lim_{n\to\infty} d(x_n,x_m) = 0.$$

Hence $\{x_n\}$ is a dqb –Cauchy sequence. Since (X,d) is complete dqb –metric space, $\{x_n\}$ dqb –converges to some $x \in X$. The subsequences $\{x_{2n}\}$ in A and $\{x_{2n-1}\}$ in B also converge to $x \in X$. As A and B are closed subspaces of X, $x \in A \cap B$. Now to prove that $x \in fx$. We consider

$$d(x,fx) \le k[d(x,x_{n+1}) + d(x_{n+1},fx)]$$
$$= kd(x,x_{n+1}) + kd(x_{n+1},fx)$$
$$\le kd(x,x_{n+1}) + kH(fx_n,fx)$$
$$= kd(x,x_{n+1}) + kH(fx,fx_n)$$
$$\le kd(x,x_{n+1}) + k\alpha(d(x,x_n))d(x,x_n)$$
$$\le kd(x,x_{n+1}) + kd(x,x_n).$$

Since above inequality is true for all $n \in N$, taking limit as $n \to \infty$, we get

$$d(x,fx) = 0.$$

As fx is closed, using lemma (1.15), we get $x \in fx$. Hence the theorem is proved.

Theorem 2.4 *Let A and B be non-empty closed subsets of a complete dqb –metric space (X,d) with coefficient k and $f: A \cup B \to CB(X)$ be a cyclic multi-valued mapping. If there exists $\alpha \in (0,1/k)$ such that $H(fx,fy) \le \alpha \max\{d(x,y),d(y,fy)\}$ for all $x \in A$ and $y \in B$. Then f has atleast one fixed point in $A \cap B$.*

Proof. Let $x_0 \in X$ be a arbitrarily fixed. Let c be such that $1 < c < \frac{1}{\alpha}$. We can choose $x_1 \in fx_0 \subset B$. By lemma (1.17), there exists $x_2 \in fx_1 \subset A$ such that

$$d(x_1,x_2) \le cH(fx_0,fx_1)$$
$$\le c\alpha\max\{d(x_0,x_1),d(x_1,fx_1)\}$$
$$\le c\alpha\max\{d(x_0,x_1),d(x_1,x_2)\}.$$

If $\max\{d(x_0,x_1),d(x_1,x_2)\} = d(x_1,x_2)$, then we get a contradiction that $1 \le c\alpha$. Therefore taking $\beta = \max\{d(x_0,x_1),d(x_1,x_0)\}$, we get $d(x_1,x_2) \le c\alpha\beta$.

Again using same argument as above, there exists $x_3 \in fx_2 \subset B$ such that

$$d(x_2,x_3) \le cH(fx_1,fx_2)$$

$$\leq c\alpha \max\{d(x_1,x_2), d(x_2, fx_2)\}$$
$$\leq c\alpha \max\{d(x_1,x_2), d(x_2,x_3)\}$$
$$\leq c\alpha \max\{c\alpha\beta, d(x_2,x_3)\}.$$

If $\max\{c\alpha\beta, d(x_2,x_3)\} = d(x_2,x_3)$, then again we get a contradiction that $1 \leq c\alpha$. Therefore $d(x_2,x_3) \leq (c\alpha)^2\beta$.

Continuing in this manner we get, for any $n \in \mathbb{N}$, $x_{n+1} \in fx_n$ such that

$$d(x_n,x_{n+1}) \leq (c\alpha)^n\beta$$
$$= s^n\beta,$$

$where \quad s = c\alpha < 1.$

Now we prove that $\{x_n\}$ is a dqb $-$Cauchy sequence. Let $m,n \in N$ such that $m > n$ and $m = n + j$ for some $j \in N$. Consider,

$$d(x_n,x_m) = d(x_n,x_{n+j})$$
$$\leq kd(x_n,x_{n+1}) + k^2 d(x_{n+1},x_{n+2}) + \cdots + k^j d(x_{n+j-1},x_{n+j})$$
$$\leq ks^n\beta + k^2 s^{n+1}\beta + \cdots + k^j s^{n+j-1}\beta$$
$$\leq [1 + ks + (ks)^2 + \cdots + (ks)^{j-1}]ks^n\beta$$
$$= \frac{1-(ks)^j}{1-ks} ks^n\beta.$$

Taking limit as $n \to \infty$ in above inequality, we get

$$\lim_{n\to\infty} d(x_n,x_m) = 0.$$

Hence $\{x_n\}$ is a dqb $-$Cauchy sequence. Since (X,d) is dqb $-$complete metric space, $\{x_n\}$ dqb $-$converges to some $x \in X$. The subsequences $\{x_{2n}\}$ in A and $\{x_{2n-1}\}$ in B also converge to $x \in X$. As A and B are closed subspaces of X, $x \in A \cap B$. Now to prove that $x \in fx$. We consider

$$d(x,fx) \leq k[d(x,x_{n+1}) + d(x_{n+1},fx)]$$
$$= kd(x,x_{n+1}) + kd(x_{n+1},fx)$$
$$\leq kd(x,x_{n+1}) + kH(fx_n,fx)$$
$$\leq kd(x,x_{n+1}) + k\alpha \max\{d(x_n,x), d(x,fx)\}.$$

Since above inequality is true for all $n \in N$, taking limit as $n \to \infty$, we get a contradiction unless

$$d(x,fx) = 0.$$

As fx is closed, using lemma (1.15), we get $x \in fx$. Hence the theorem is proved.

Theorem 2.5 *Let A and B be non-empty closed subsets of a complete dqb $-$metric space (X,d) with coefficient k and $f: A \cup B \to CB(X)$ be a cyclic multi-valued mapping. If there exists $\alpha \in (0,1/k)$ such that $H(fx,fy) \leq \alpha \max\{d(x,fx), d(y,fy)\}$ for all $x \in A$ and $y \in B$. Then f has atleast one fixed point in $A \cap B$.*

Proof. Let $x_0 \in X$ be a arbitrarily fixed. Let c be such that $1 < c < \frac{1}{\alpha}$. We can choose $x_1 \in fx_0 \subset B$. By lemma (1.17), there exists $x_2 \in fx_1 \subset A$ such that

$$d(x_1,x_2) \leq cH(fx_0,fx_1)$$
$$\leq c\alpha \max\{d(x_0,fx_0), d(x_1,fx_1)\}$$
$$\leq c\alpha \max\{d(x_0,x_1), d(x_1,x_2)\}.$$

If $\max\{d(x_0,x_1), d(x_1,x_2)\} = d(x_1,x_2)\}$, then we get a contradiction that $1 \leq c\alpha$. Therefore we get $d(x_1,x_2) \leq c\alpha d(x_1,x_0)$.

Again using same argument as above, there exists $x_3 \in fx_2 \subset B$ such that

$$d(x_2,x_3) \leq cH(fx_1,fx_2)$$
$$\leq c\alpha \max\{d(x_1,fx_1), d(x_2,fx_2)\}$$

$$\leq c\alpha \max\{d(x_1,x_2), d(x_2,x_3)\}$$
$$\leq c\alpha \max\{c\alpha\beta, d(x_2,x_3)\}.$$

If $\max\{c\alpha\beta, d(x_2,x_3)\} = d(x_2,x_3)\}$, then again we get a contradiction that $1 \leq c\alpha$. Therefore $d(x_2,x_3) \leq (c\alpha)^2\beta$.

Continuing in this manner we get, for any $n \in \mathbb{N}$, $x_{n+1} \in fx_n$ such that

$$d(x_n,x_{n+1}) \leq (c\alpha)^n\beta$$
$$= s^n\beta,$$

$where \quad s = c\alpha < 1.$

Now we prove that $\{x_n\}$ is a dqb −Cauchy sequence. Let $m,n \in N$ such that $m > n$ and $m = n + j$ for some $j \in N$. Consider,

$$d(x_n,x_m) = d(x_n,x_{n+j})$$
$$\leq kd(x_n,x_{n+1}) + k^2d(x_{n+1},x_{n+2}) + \cdots + k^jd(x_{n+j-1},x_{n+j})$$
$$\leq ks^n\beta + k^2s^{n+1}\beta + \cdots + k^js^{n+j-1}\beta$$
$$\leq [1 + ks + (ks)^2 + \cdots + (ks)^{j-1}]ks^n\beta$$
$$= \frac{1-(ks)^j}{1-ks}ks^n\beta.$$

Taking limit as $n \to \infty$ in above inequality, we get

$$\lim_{n\to\infty} d(x_n,x_m) = 0.$$

Hence $\{x_n\}$ is a dqb −Cauchy sequence. Since (X,d) is dqb −complete metric space, $\{x_n\}$ dqb −converges to some $x \in X$. The subsequences $\{x_{2n}\}$ in A and $\{x_{2n-1}\}$ in B also converge to $x \in X$. As A and B are closed subspaces of X, $x \in A \cap B$. Now to prove that $x \in fx$. We consider

$$d(x,fx) \leq k[d(x,x_{n+1}) + d(x_{n+1},fx)]$$
$$= kd(x,x_{n+1}) + kd(x_{n+1},fx)$$
$$\leq kd(x,x_{n+1}) + kH(fx_n,fx)$$
$$\leq kd(x,x_{n+1}) + k\alpha \max\{d(x_n,fx_n), d(x,fx)\}$$
$$\leq kd(x,x_{n+1}) + k\alpha \max\{d(x_n,x_{n+1}), d(x,fx)\}.$$

Since above inequality is true for all $n \in N$, taking limit as $n \to \infty$, we get a contradiction unless

$$d(x,fx) = 0.$$

As fx is closed, using lemma (1.15), we get $x \in fx$. Hence the theorem is proved.

Theorem 2.6 *Let (X,d) be a complete dqb −metric space with coefficient $k \geq 1$. Let $T: X \to X$ and $f: A \cup B \to A \cup B$ be self-mappings such that T is continuous, one-one and f is continuous T −Banach cyclic multivalued contraction with $k\alpha \leq 1$. Then f has atleast one fixed point in $A \cap B$.*

Proof. Let $x_0 \in X$ be a arbitrarily fixed. Let c be such that $1 < c < \frac{1}{\alpha}$. We can choose $x_1 \in fx_0 \subset B$. By lemma (1.17), there exists $x_2 \in fx_1 \subset A$ such that

$$d(Tx_1,Tx_2) \leq cH(Tfx_0,Tfx_1)$$

Since f is T −Banach cyclic multivalued contraction with $k\alpha \leq 1$, we have,

$$d(Tx_1,Tx_2) \leq c\alpha d(Tx_0,Tx_1).$$

Again using same argument as above, there exists $x_3 \in fx_2 \subset B$ such that

$$d(Tx_2,Tx_3) \leq cH(Tfx_1,Tfx_2)$$
$$\leq c\alpha d(Tx_1,Tx_2)$$
$$\leq c^2\alpha^2 d(Tx_0,Tx_1).$$

Thus we can obtain for any $n \in N$, $x_{n+1} \in fx_n$ such that

$$d(Tx_n,Tx_{n+1}) \le c^n\alpha^n d(Tx_0,Tx_1)$$
$$= s^n d(Tx_0,Tx_1), where \quad s = c\alpha < 1.$$

Now we prove that $\{Tx_n\}$ is a dqb −Cauchy sequence. Let $m,n \in N$ such that $m > n$ and $m = n + j$ for some $j \in N$. Consider,

$$d(Tx_n,Tx_m) = d(Tx_n,Tx_{n+j})$$
$$\le kd(Tx_n,Tx_{n+1}) + k^2d(Tx_{n+1},Tx_{n+2}) + \cdots + k^jd(Tx_{n+j-1},Tx_{n+j})$$
$$\le ks^nd(Tx_0,Tx_1) + k^2s^{n+1}d(Tx_0,Tx_1) + \cdots + k^js^{n+j-1}d(Tx_0,Tx_1)$$
$$\le [1 + ks + (ks)^2 + \cdots + (ks)^{j-1}]ks^nd(Tx_0,Tx_1)$$
$$= \frac{1-(ks)^j}{1-ks}ks^nd(Tx_0,Tx_1).$$

Taking limit as $n \to \infty$ in above inequality, we get

$$\lim_{n\to\infty} d(Tx_n,Tx_m) = 0.$$

Hence $\{Tx_n\}$ is a dqb −Cauchy sequence. Since (X,d) is dqb −complete metric space, $\{Tx_n\}$ dqb −converges to some $x \in X$. The subsequences $\{Tx_{2n}\}$ in A and $\{Tx_{2n-1}\}$ in B also converge to $x \in X$. As A and B are closed subspaces of X, $x \in A \cap B$.

$$d(Tx,Tfx) \le k[d(Tx,Tx_{n+1}) + d(Tx_{n+1},Tfx)]$$
$$= kd(Tx,Tx_{n+1}) + kd(Tx_{n+1},Tfx)$$
$$\le kd(Tx,Tx_{n+1}) + kH(Tfx_n,Tfx)$$
$$= kd(Tx,Tx_{n+1}) + kH(Tfx,Tfx_n)$$
$$\le kd(Tx,Tx_{n+1}) + k\alpha d(Tx,Tx_n)$$
$$\le kd(Tx,Tx_{n+1}) + kd(Tx,Tx_n).$$

Since above inequality is true for all $n \in N$, taking limit as $n \to \infty$, we get a contradiction unless

$$d(Tx,Tfx) = 0.$$

As Tfx is closed, using lemma (1.15), we get $Tx \in Tfx$ that is $x \in fx$. Hence the theorem is proved.

Theorem 2.7 *Let (X,d) be a complete dqb −metric space with coefficient $k \ge 1$. Let $T: X \to X$ and $f: A \cup B \to A \cup B$ be self-mappings such that T is continuous,one-one and f is continuous T −Kannan cyclic multivalued contraction with $k\alpha \le 1/2$. Then f has atleast one fixed point in $A \cap B$.*

Proof. Let $x_0 \in X$ be a arbitrarily fixed. Let c be such that $1 < c < \frac{1}{2\alpha}$. We can choose $x_1 \in fx_0 \subset B$. By lemma (1.17), there exists $x_2 \in fx_1 \subset A$ such that

$$d(Tx_1,Tx_2) \le cH(Tfx_0,Tfx_1)$$

Since f is T −Kannan cyclic multivalued contraction with $k\alpha \le 1$, we have,

$$d(Tx_1,Tx_2) \le c\alpha[d(Tx_0,Tfx_0) + d(Tx_1,Tfx_1)]$$
$$\le \frac{c\alpha}{1-c\alpha}d(Tx_0,Tx_1).$$

Again using same argument as above, there exists $x_3 \in fx_2 \subset B$ such that

$$d(Tx_2,Tx_3) \le cH(Tfx_1,Tfx_2)$$
$$\le c\alpha[d(Tx_1,Tfx_1) + d(Tx_2,Tfx_2)]$$
$$\le \frac{c\alpha}{1-c\alpha}d(Tx_1,Tx_2)$$
$$\le \lambda^2d(Tx_0,Tx_1), \quad where \quad \lambda = \frac{c\alpha}{1-c\alpha} < 1.$$

Thus we can obtain for any $n \in N$, $x_{n+1} \in fx_n$ such that

$$d(Tx_n,Tx_{n+1}) \le \lambda^nd(Tx_0,Tx_1).$$

Now we prove that $\{Tx_n\}$ is a dqb −Cauchy sequence. Let $m,n \in N$ such that $m > n$ and

$m = n + j$ for some $j \in N$. Consider,

$$\begin{aligned} d(Tx_n, Tx_m) &= d(Tx_n, Tx_{n+j}) \\ &\leq kd(Tx_n, Tx_{n+1}) + k^2 d(Tx_{n+1}, Tx_{n+2}) + \cdots + k^j d(Tx_{n+j-1}, Tx_{n+j}) \\ &\leq k\lambda^n d(Tx_0, Tx_1) + k^2\lambda^{n+1} d(Tx_0, Tx_1) + \cdots + k^j \lambda^{n+j-1} d(Tx_0, Tx_1) \\ &\leq [1 + k\lambda + (k\lambda)^2 + \cdots + (k\lambda)^{j-1}] k\lambda^n d(Tx_0, Tx_1) \\ &= \frac{1-(k\lambda)^j}{1-k\lambda} k\lambda^n d(Tx_0, Tx_1). \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we get

$$\lim_{n\to\infty} d(Tx_n, Tx_m) = 0$$

. Hence $\{Tx_n\}$ is a dqb −Cauchy sequence. Since (X, d) is dqb −complete metric space, $\{Tx_n\}$ dqb −converges to some $x \in X$. The subsequences $\{Tx_{2n}\}$ in A and $\{Tx_{2n-1}\}$ in B also converge to $x \in X$. As A and B are closed subspaces of X, $x \in A \cap B$.

$$\begin{aligned} d(Tx, Tfx) &\leq k[d(Tx, Tx_{n+1}) + d(Tx_{n+1}, Tfx)] \\ &= kd(Tx, Tx_{n+1}) + kd(Tx_{n+1}, Tfx) \\ &\leq kd(Tx, Tx_{n+1}) + kH(Tfx_n, Tfx) \\ &= kd(Tx, Tx_{n+1}) + kH(Tfx, Tfx_n) \\ &\leq kd(Tx, Tx_{n+1}) + k\alpha[d(Tx, Tfx) + d(Tx_n, Tfx_n)] \\ &\leq kd(Tx, Tx_{n+1}) + kd(Tx, Tfx) + kd(Tx_n, Tx_{n+1}) \\ &\leq \frac{k}{1-k}[d(Tx, Tx_{n+1}) + d(Tx_n, Tx_{n+1})]. \end{aligned}$$

Since above inequality is true for all $n \in N$, taking limit as $n \to \infty$, we get a contradiction unless

$$d(Tx, Tfx) = 0.$$

As Tfx is closed, using lemma (1.15), we get $Tx \in Tfx$ that is $x \in fx$. Hence the theorem is proved.

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