



Some Products of Γ – Reset Fuzzy Automata

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Abstract In this paper, we study the characterization of Γ -reset fuzzy automata through their products. We prove that the cascade product, restricted cascade product, wreath product, sum and Cartesian composition of Γ -reset fuzzy automata are Γ -reset fuzzy automata. These results extend the existing results on direct product and restricted direct product to more general product constructions in fuzzy automata theory.

Keyword Γ -reset fuzzy automaton, cascade product, restricted cascade product, wreath product, sum, strong homomorphism.

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1. INTRODUCTION

Fuzzy sets were introduced by Zadeh in 1965 [8]. Subsequently, the concept of fuzzy automata was formulated by Wee in 1967 [5, 6]. Within the study of directable and synchronizable fuzzy automata, Karthikeyan [1] introduced the notion of Γ -reset fuzzy automata, which was further investigated and extended by Mohanarao and Karthikeyan in 2021 [4]. This framework generalizes the concept of reset (synchronizing) words from classical automata theory to parameterized Γ -resetting behaviors in fuzzy transition systems.

The algebraic study of machine compositions was extended to fuzzy finite state machines by Malik et al. [3], who introduced the cascade product and wreath product along with the associated notions of transformation semigroups and coverings. Further enriching this structural foundation, Kumbhojkar and Chaudhari [2] systematically introduced the notions of sum, and restricted products for fuzzy finite state machines, establishing fundamental results on their covering relations. Later, product constructions were expanded to specialized classes such as switchboard state machines, including the restricted cascade product investigated in intuitionistic and general fuzzy settings [7].

In this paper, we study the algebraic closure properties of Γ -reset fuzzy automata under various product operations. In particular, we prove that the cascade product, restricted cascade product, wreath product and sum of Γ –reset fuzzy automata are also Γ -reset fuzzy automata.

2. PRELIMINARIES

2.1. Fuzzy Automata.

Definition 2.1. [5] A fuzzy automaton is a triplet $F = (Q, X, \eta)$ where, Q is a non empty finite set of states $\{q_1, q_2, \dots, q_n\}$, X is the input set of alphabets and $\eta : Q \times X \times Q \rightarrow [0, 1]$ is the fuzzy transition function defined by $\eta(q_1, a, q_2) = \nu$ where $0 \leq \nu \leq 1$ and $a \in X, q_1, q_2 \in Q$

The extension of η to strings is defined as,

$$\eta^*(q_1, \lambda, q_2) = \begin{cases} 1 & \text{if } q_1 = q_2 \\ 0 & \text{if } q_1 \neq q_2 \end{cases}$$

where λ is the empty string and for $w = ax$ where $a \in X$ and $x \in X^*$

$$\eta^*(q_1, ax, q_2) = \bigvee_{r \in Q} \{ \eta(q_1, a, r) \wedge \eta^*(r, x, q_2) \}$$

2.2. Γ -Reset Fuzzy Automaton.

Definition 2.2. [1, 4] Let $F = (Q, X, \eta)$ be a fuzzy automaton, for a given threshold $\Gamma \in (0, 1]$, F is called a Γ -reset fuzzy automaton at state $q_1 \in Q$ if there exists a string $w \in X^*$ such that

$$\eta^*(q_2, w, q_1) \geq \Gamma \quad \text{for all } q_2 \in Q$$

The string w is called a Γ -reset string (or Γ -synchronizing word) for F with respect to state q_1 .

Example 2.3. [Γ -Reset Fuzzy Automaton] Let $F = (Q, X, \eta)$ be a fuzzy automaton, where $Q = \{q_1, q_2, q_3\}$, $X = \{a, b\}$ and $\Gamma = 0.7$, The fuzzy transition function $\eta : Q \times X \times Q \rightarrow [0, 1]$ is represented by the fuzzy transition matrices M_a and M_b , where the entry in the i -th row and j -th column denotes $(M_x)_{ij} = \eta(q_i, x, q_j)$

$$M_a = \begin{pmatrix} 0.8 & 0.2 & 0.1 \\ 0.7 & 0.3 & 0.2 \\ 0.1 & 0.9 & 0.2 \end{pmatrix}, \quad M_b = \begin{pmatrix} 0.85 & 0.1 & 0.1 \\ 0.80 & 0.2 & 0.1 \\ 0.75 & 0.1 & 0.3 \end{pmatrix}$$

Composition Rule (Max-Min Composition). For any string $w = uv \in X^*$, the extended transition function is evaluated using the max-min matrix composition $M_{uv} = M_u \circ M_v$, defined by

$$(M_u \circ M_v)_{ij} = \bigvee_{k=1}^3 [(M_u)_{ik} \wedge (M_v)_{kj}] = \max_{1 \leq k \leq 3} \min((M_u)_{ik}, (M_v)_{kj})$$

Computation for String $w = ab$. To determine whether $w = ab$ resets all states into the target state q_1 , we compute $\eta^*(q_i, ab, q_1) = (M_a \circ M_b)_{i1}$ for each $q_i \in Q$

$$\begin{aligned} \eta^*(q_1, ab, q_1) &= \max \{ \min(0.8, 0.85), \min(0.2, 0.80), \min(0.1, 0.75) \} \\ &= \max \{ 0.8, 0.2, 0.1 \} = \mathbf{0.8} \end{aligned}$$

$$\begin{aligned} \eta^*(q_2, ab, q_1) &= \max \{ \min(0.7, 0.85), \min(0.3, 0.80), \min(0.2, 0.75) \} \\ &= \max \{ 0.7, 0.3, 0.2 \} = \mathbf{0.7} \end{aligned}$$

$$\begin{aligned} \eta^*(q_3, ab, q_1) &= \max \{ \min(0.1, 0.85), \min(0.9, 0.80), \min(0.2, 0.75) \} \\ &= \max \{ 0.1, 0.8, 0.2 \} = \mathbf{0.8} \end{aligned}$$

Verification. Evaluating the condition $\eta^*(q, ab, q_1) \geq \Gamma$ for all $q \in Q$ with $\Gamma = 0.7$

$$\eta^*(q_1, ab, q_1) = 0.8 \geq 0.7,$$

$$\eta^*(q_2, ab, q_1) = 0.7 \geq 0.7,$$

$$\eta^*(q_3, ab, q_1) = 0.8 \geq 0.7.$$

Since $\eta^*(q, ab, q_1) \geq 0.7$ for every $q \in Q$, the automaton F is a **0.7-reset fuzzy automaton at state q_1** , and the string $w = ab$ is a **0.7-reset string**.

3. PRODUCTS OF Γ -RESET FUZZY AUTOMATA

3.1. Cascade Product.

Definition 3.1. [3] Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata. The **cascade product** of F_1 and F_2 is denoted by $F_1 \omega F_2 = (Q_1 \times Q_2, X_2, \eta_1 \omega \eta_2)$ where $\omega : Q_2 \times X_2 \rightarrow X_1$ is a mapping and $(\eta_1 \omega \eta_2) : (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \rightarrow [0, 1]$ is defined as

$$(\eta_1 \omega \eta_2)((q_1, q_2), a_2, (p_1, p_2)) = \eta_1(q_1, \omega(q_2, a_2), p_1) \wedge \eta_2(q_2, a_2, p_2)$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $a_2 \in X_2$.

Lemma 3.2. Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata. Let $F_1 \omega F_2$ be the cascade product of F_1 and F_2 . Then for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, and for all $w_2 \in X_2^*$

$$(\eta_1 \omega \eta_2)^*((q_1, q_2), w_2, (p_1, p_2)) = \eta_1^*(q_1, \omega^*(q_2, w_2), p_1) \wedge \eta_2^*(q_2, w_2, p_2)$$

Proof. We prove by induction on length of w_2 denoted by $|w_2| = n$.

Case-I Suppose that $n = 0$ then $w_2 = \lambda$. Then by Definition (3.1)

$$(\eta_1 \omega \eta_2)^*((q_1, q_2), \lambda, (p_1, p_2)) = \begin{cases} 1 & \text{if } (q_1, q_2) = (p_1, p_2) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\eta_1^*(q_1, \omega^*(q_2, \lambda), p_1) \wedge \eta_2^*(q_2, \lambda, p_2) = \eta_1^*(q_1, \lambda, p_1) \wedge \eta_2^*(q_2, \lambda, p_2) = \begin{cases} 1 & \text{if } (q_1, q_2) = (p_1, p_2) \\ 0 & \text{otherwise} \end{cases}$$

The result holds for $n = 0$.

Case-II Suppose that $n = 1$. Let $w_2 = a_2 \in X_2$. Then

$$\begin{aligned} (\eta_1 \omega \eta_2)^*((q_1, q_2), a_2, (p_1, p_2)) &= \eta_1(q_1, \omega(q_2, a_2), p_1) \wedge \eta_2(q_2, a_2, p_2) \\ &= \eta_1^*(q_1, \omega(q_2, a_2), p_1) \wedge \eta_2^*(q_2, a_2, p_2) \\ &= \eta_1^*(q_1, \omega^*(q_2, a_2), p_1) \wedge \eta_2^*(q_2, a_2, p_2) \end{aligned}$$

Case-III Suppose the result is true for all strings of length $n - 1$. Let $w_2 = a_2 v_2$ where $a_2 \in X_2$, $v_2 \in X_2^*$, and $|v_2| = n - 1$. Then,

$$\begin{aligned} &(\eta_1 \omega \eta_2)^*((q_1, q_2), a_2 v_2, (p_1, p_2)) \\ &= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} \{(\eta_1 \omega \eta_2)((q_1, q_2), a_2, (r_1, r_2)) \wedge (\eta_1 \omega \eta_2)^*((r_1, r_2), v_2, (p_1, p_2))\} \\ &= \bigvee_{r_1 \in Q_1, r_2 \in Q_2} \{\eta_1(q_1, \omega(q_2, a_2), r_1) \wedge \eta_2(q_2, a_2, r_2) \wedge \eta_1^*(r_1, \omega^*(r_2, v_2), p_1) \wedge \eta_2^*(r_2, v_2, p_2)\} \\ &= \left(\bigvee_{r_1 \in Q_1} \{\eta_1(q_1, \omega(q_2, a_2), r_1) \wedge \eta_1^*(r_1, \omega^*(r_2, v_2), p_1)\} \right) \wedge \left(\bigvee_{r_2 \in Q_2} \{\eta_2(q_2, a_2, r_2) \wedge \eta_2^*(r_2, v_2, p_2)\} \right) \\ &= \eta_1^*(q_1, \omega(q_2, a_2) \cdot \omega^*(r_2, v_2), p_1) \wedge \eta_2^*(q_2, a_2 v_2, p_2) \end{aligned}$$

$$\text{Set } \omega(q_2, a_2) \cdot \omega^*(r_2, v_2) = \omega^*(q_2, a_2 v_2)$$

$$= \eta_1^*(q_1, \omega^*(q_2, a_2 v_2), p_1) \wedge \eta_2^*(q_2, w_2, p_2)$$

Therefore, the result holds for all $w_2 \in X_2^*$.

Theorem 3.3. *The cascade product of Γ -reset fuzzy automata is a Γ -reset fuzzy automaton.*

Proof. Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be Γ_k -reset fuzzy automata. Then the cascade product of F_1 and F_2 is given by $F_1 \omega F_2 = (Q_1 \times Q_2, X_2, \eta_1 \omega \eta_2)$. Since F_1 and F_2 are Γ -reset fuzzy automata, there exist Γ -reset strings $w_1 \in X_1^*$, $w_2 \in X_2^*$ and states $p_1 \in Q_1$, $p_2 \in Q_2$ such that $\eta_1^*(q_1, w_1, p_1) \geq \Gamma_1$ and $\eta_2^*(q_2, w_2, p_2) \geq \Gamma_2$ for all $q_1 \in Q_1$ and $q_2 \in Q_2$. By Lemma(3.2)

$$(\eta_1 \omega \eta_2)^*((q_1, q_2), w_2, (p_1, p_2)) = \eta_1^*(q_1, \omega^*(q_2, w_2), p_1) \wedge \eta_2^*(q_2, w_2, p_2)$$

Since $\omega^*(q_2, w_2) \in X_1^*$ and F_1 is Γ -reset, we have,

$$\eta_1^*(q_1, \omega^*(q_2, w_2), p_1) \geq \Gamma_1$$

Also, since F_2 is Γ -reset

$$\eta_2^*(q_2, w_2, p_2) \geq \Gamma_2$$

Therefore,

$$(\eta_1 \omega \eta_2)^*((q_1, q_2), w_2, (p_1, p_2)) \geq \Gamma_1 \wedge \Gamma_2 = \Gamma$$

Hence, $F_1 \omega F_2$ is a Γ -reset fuzzy automaton.

3.2. Restricted Cascade Product.

Definition 3.4. [3] Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata. The **restricted cascade product** of F_1 and F_2 is denoted by $F_1 \varpi F_2 = (Q_1 \times Q_2, X_2, \eta_1 \varpi \eta_2)$ where $\varpi : X_2 \rightarrow X_1$ is a mapping and $(\eta_1 \varpi \eta_2) : (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \rightarrow [0, 1]$ is defined as $(\eta_1 \varpi \eta_2)((q_1, q_2), a_2, (p_1, p_2)) = \eta_1(q_1, \varpi(a_2), p_1) \wedge \eta_2(q_2, a_2, p_2)$ for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $a_2 \in X_2$.

Lemma 3.5. *Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata. Let $F_1 \varpi F_2$ be the restricted cascade product of F_1 and F_2 . Then for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, and for all $w_2 \in X_2^*$*

$$(\eta_1 \varpi \eta_2)^*((q_1, q_2), w_2, (p_1, p_2)) = \eta_1^*(q_1, \varpi^*(w_2), p_1) \wedge \eta_2^*(q_2, w_2, p_2)$$

Proof. The proof follows by induction on $|w_2| = n$.

Case-I We prove the result is true for $n = 1$. Let $w_2 = a_2 \in X_2$. Then by Definition (3.4)

$$\begin{aligned} (\eta_1 \varpi \eta_2)^*((q_1, q_2), a_2, (p_1, p_2)) &= \eta_1(q_1, \varpi(a_2), p_1) \wedge \eta_2(q_2, a_2, p_2) \\ &= \eta_1(q_1, \varpi(a_2), p_1) \wedge \eta_2(q_2, a_2, p_2) \\ &= \eta_1^*(q_1, \varpi^*(a_2), p_1) \wedge \eta_2^*(q_2, a_2, p_2) \end{aligned}$$

Case-II Suppose the result is true for all strings of length $n - 1$. Let $w_2 = a_2 v_2$ where $a_2 \in X_2$, $v_2 \in X_2^*$, and $|v_2| = n - 1$. Then,

$$\begin{aligned} &(\eta_1 \varpi \eta_2)^*((q_1, q_2), a_2 v_2, (p_1, p_2)) \\ &= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} \{(\eta_1 \varpi \eta_2)((q_1, q_2), a_2, (r_1, r_2)) \wedge (\eta_1 \varpi \eta_2)^*((r_1, r_2), v_2, (p_1, p_2))\} \\ &= \bigvee_{r_1 \in Q_1, r_2 \in Q_2} \{\eta_1(q_1, \varpi(a_2), r_1) \wedge \eta_2(q_2, a_2, r_2) \wedge \eta_1^*(r_1, \varpi^*(v_2), p_1) \wedge \eta_2^*(r_2, v_2, p_2)\} \\ &= \eta_1^*(q_1, \varpi(a_2) \cdot \varpi^*(v_2), p_1) \wedge \eta_2^*(q_2, a_2 v_2, p_2) \end{aligned}$$

$$\text{Set } \varpi(a_2) \cdot \varpi^*(v_2) = \varpi^*(a_2 v_2)$$

$$\begin{aligned} &= \eta_1^*(q_1, \varpi^*(a_2 v_2), p_1) \wedge \eta_2^*(q_2, w_2, p_2) \\ &= \eta_1^*(q_1, \varpi^*(w_2), p_1) \wedge \eta_2^*(q_2, w_2, p_2) \end{aligned}$$

Thus, the result holds for all $w_2 \in X_2^*$.

Theorem 3.6. *The restricted cascade product of Γ -reset fuzzy automata is a Γ -reset fuzzy automaton.*

Proof. Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be Γ_k -reset fuzzy automata. Then the restricted cascade product of F_1 and F_2 is given by $F_1 \varpi F_2 = (Q_1 \times Q_2, X_2, \eta_1 \varpi \eta_2)$. Since F_1 and F_2 are Γ -reset fuzzy automata, there exist Γ -reset strings $w_1 \in X_1^*$, $w_2 \in X_2^*$ and states $p_1 \in Q_1$, $p_2 \in Q_2$ such that $\eta_1^*(q_1, w_1, p_1) \geq \Gamma_1$ and $\eta_2^*(q_2, w_2, p_2) \geq \Gamma_2$ for all $q_1 \in Q_1$ and $q_2 \in Q_2$. By Lemma (3.5)

$$(\eta_1 \varpi \eta_2)^*((q_1, q_2), w_2, (p_1, p_2)) = \eta_1^*(q_1, \varpi^*(w_2), p_1) \wedge \eta_2^*(q_2, w_2, p_2)$$

Since $\varpi^*(w_2) \in X_1^*$ and F_1 is Γ -reset

$$\eta_1^*(q_1, \varpi^*(w_2), p_1) \geq \Gamma_1$$

Also, since F_2 is Γ -reset

$$\eta_2^*(q_2, w_2, p_2) \geq \Gamma_2$$

Therefore,

$$(\eta_1 \varpi \eta_2)^*((q_1, q_2), w_2, (p_1, p_2)) \geq \Gamma_1 \wedge \Gamma_2 = \Gamma$$

Hence, $F_1 \varpi F_2$ is a Γ -reset fuzzy automaton.

3.3. Wreath Product.

Definition 3.7. [3] Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata. The wreath product of F_1 and F_2 is denoted by $F_1 \circ F_2 = (Q_1 \times Q_2, X_2, \eta_1 \circ \eta_2)$ where $X_1^{Q_2} = \{f : Q_2 \rightarrow X_1\}$ and $(\eta_1 \circ \eta_2) : (Q_1 \times Q_2) \times (X_1^{Q_2} \times X_2) \times (Q_1 \times Q_2) \rightarrow [0, 1]$ defined by $(\eta_1 \circ \eta_2)((q_1, q_2), (f, a_2), (p_1, p_2)) = \eta_1(q_1, f(q_2), p_1) \wedge \eta_2(q_2, a_2, p_2)$ for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, $a_2 \in X_2$, and $f \in X_1^{Q_2}$.

Lemma 3.8. *Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata, and let $F_1 \circ F_2$ be the wreath product of F_1 and F_2 . Then for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$, $f \in X_1^{Q_2}$ and $w_2 \in X_2^*$,*

$$(\eta_1 \circ \eta_2)^*((q_1, q_2), (f, u_2), (p_1, p_2)) = \eta_1^*(q_1, f(q_2), p_1) \wedge \eta_2^*(q_2, u_2, p_2)$$

Proof. The proof follows by induction on $|w_2| = n$.

Case-I We prove the result is true for $n = 1$. Let $f \in X_1^{Q_2}$ and $w_2 = a_2 \in X_2$. Then by Definition (3.7)

$$\begin{aligned} (\eta_1 \circ \eta_2)^*((q_1, q_2), (f, a_2), (p_1, p_2)) &= (\eta_1 \circ \eta_2)((q_1, q_2), (f, a_2), (p_1, p_2)) \\ &= \eta_1(q_1, f(q_2), p_1) \wedge \eta_2(q_2, a_2, p_2) \\ &= \eta_1^*(q_1, f(q_2), p_1) \wedge \eta_2^*(q_2, a_2, p_2) \end{aligned}$$

Case-II Suppose the result is true for all strings of length $n - 1$. Let $f \in X_1^{Q_2}$, $w_2 = a_2 v_2 \in X_2$, where $a_2 \in X_2$, $v_2 \in X_2^*$ and $|v_2| = n - 1$. Then,

$$\begin{aligned} & (\eta_1 \circ \eta_2)^*((q_1, q_2), (f, a_2 v_2), (p_1, p_2)) \\ &= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} \{(\eta_1 \circ \eta_2)((q_1, q_2), (f, a_2), (r_1, r_2)) \wedge (\eta_1 \omega \eta_2)^*((r_1, r_2), (f, v_2), (p_1, p_2))\} \\ &= \bigvee_{r_1 \in Q_1, r_2 \in Q_2} \{\eta_1(q_1, f(q_2), r_1) \wedge \eta_2(q_2, a_2, r_2) \wedge \eta_1^*(r_1, f(r_2), p_1) \wedge \eta_2^*(r_2, v_2, p_2)\} \\ &= \left(\bigvee_{r_1 \in Q_1} \{\eta_1(q_1, f(q_2), r_1) \wedge \eta_1^*(r_1, f(r_2), p_1)\} \right) \wedge \left(\bigvee_{r_2 \in Q_2} \{\eta_2(q_2, a_2, r_2) \wedge \eta_2^*(r_2, v_2, p_2)\} \right) \\ &= \eta_1^*(q_1, f(q_2) \cdot f(r_2), p_1) \wedge \eta_2^*(q_2, a_2 v_2, p_2) \end{aligned}$$

Set $f(q_2) \cdot f(r_2) = f(q_2)$

$$= \eta_1^*(q_1, f(q_2), p_1) \wedge \eta_2^*(q_2, v_2, p_2)$$

Thus, the result holds for all $w_2 \in X_2^*$.

Theorem 3.9. *The wreath product of Γ -reset fuzzy automata is a Γ -reset fuzzy automaton.*

Proof. Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be Γ_k -reset fuzzy automata. Then the wreath product of F_1 and F_2 is given by,

$$F_1 \circ F_2 = (Q_1 \times Q_2, X_1^{Q_2} \times X_2, \eta_1 \circ \eta_2)$$

Since F_1 and F_2 are Γ -reset fuzzy automata, there exist Γ -reset strings $w_1 \in X_1^*$, $w_2 \in X_2^*$ and states $p_1 \in Q_1$, $p_2 \in Q_2$ such that,

$$\eta_1^*(q_1, w_1, p_1) \geq \Gamma_1 \quad \text{and} \quad \eta_2^*(q_2, w_2, p_2) \geq \Gamma_2$$

for all $q_1 \in Q_1$ and $q_2 \in Q_2$. Now, choose $\alpha \in (X_1^{Q_2} \times X_2)^*$ such that $\pi_2(\alpha) = w_2$ and $f^*(q_2, \alpha) = w_1$. Such α can be constructed by selecting functions $f_i \in X_1^{Q_2}$ appropriately along the path determined by w_2 . By Lemma (3.8)

$$\begin{aligned} (\eta_1 \circ \eta_2)^*((q_1, q_2), (f, w_2), (p_1, p_2)) &= \eta_1^*(q_1, f(q_2), p_1) \wedge \eta_2^*(q_2, w_2, p_2) \\ &= \eta_1^*(q_1, f(q_2), p_1) \wedge \eta_2^*(q_2, w_2, p_2) \\ &\geq \Gamma_1 \wedge \Gamma_2 = \Gamma \end{aligned}$$

Therefore, $F_1 \circ F_2$ is a Γ -reset fuzzy automaton.

3.4. Sum.

Definition 3.10. [2] Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata with $Q_1 \cap Q_2 = \emptyset$. The **sum** of F_1 and F_2 is denoted by $F_1 + F_2 = (Q_1 \cup Q_2, X_1 \cup X_2, \eta_1 + \eta_2)$, where the fuzzy transition relation

$$(\eta_1 + \eta_2) : (Q_1 \cup Q_2) \times (X_1 \cup X_2) \times (Q_1 \cup Q_2) \rightarrow [0, 1]$$

is defined by

$$(\eta_1 + \eta_2)(q, x, p) = \begin{cases} \eta_1(q, x, p) & \text{if } q, p \in Q_1, \text{ and } x \in X_1, \\ \eta_2(q, x, p) & \text{if } q, p \in Q_2, \text{ and } x \in X_2, \\ 0 & \text{otherwise} \end{cases}$$

for all $q, p \in Q_1 \cup Q_2$ and $x \in X_1 \cup X_2$.

Lemma 3.11. Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata with $Q_1 \cap Q_2 = \emptyset$, and let $F_1 + F_2$ be the sum of F_1 and F_2 . Then for all $q, p \in Q_1 \cup Q_2$ and $w \in (X_1 \cup X_2)^*$,

$$(\eta_1 + \eta_2)^*(q, w, p) = \begin{cases} \eta_1^*(q, w, p) & \text{if } q, p \in Q_1, \text{ and } w \in X_1, \\ \eta_2^*(q, w, p) & \text{if } q, p \in Q_2, \text{ and } w \in X_2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof follows by induction on $|w| = n$.

Case-I Let $n = 1$ and $w = a \in X$. Then by Definition (3.10),

$$(\eta_1 + \eta_2)^*(q, a, p) = (\eta_1 + \eta_2)(q, a, p) = \begin{cases} \eta_1(q, a, p) = \eta_1^*(q, a, p) & \text{if } q, p \in Q_1, \text{ and } a \in X_1, \\ \eta_2(q, a, p) = \eta_2^*(q, a, p) & \text{if } q, p \in Q_2, \text{ and } a \in X_2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the result holds for $n = 1$.

Case-II Assume that the result is true for all strings of length $n - 1$. Let $w = av \in X^*$, where $a \in X$, $v \in X^*$ with $|v| = n - 1$. By the definition of the extended transition function,

$$(\eta_1 + \eta_2)^*(q, av, p) = \bigvee_{r \in Q_1 \cup Q_2} \{(\eta_1 + \eta_2)(q, a, r) \wedge (\eta_1 + \eta_2)^*(r, v, p)\}.$$

Since $Q_1 \cap Q_2 = \emptyset$, we consider the following three subcases:

- (1) Let $q, p \in Q_1$. If $r \in Q_2$, then $(\eta_1 + \eta_2)(q, a, r) = 0$ since $q \in Q_1$ and $r \in Q_2$. Thus, non-zero terms can only occur when $r \in Q_1$. Using the induction hypothesis for $r, p \in Q_1$:

$$\begin{aligned} (\eta_1 + \eta_2)^*(q, av, p) &= \bigvee_{r \in Q_1} \{(\eta_1 + \eta_2)(q, a, r) \wedge (\eta_1 + \eta_2)^*(r, v, p)\} \\ &= \bigvee_{r \in Q_1} \{\eta_1(q, a, r) \wedge \eta_1^*(r, v, p)\} \\ &= \eta_1^*(q, av, p). \end{aligned}$$

- (2) Let $q, p \in Q_2$. Similarly, if $r \in Q_1$, then $(\eta_1 + \eta_2)(q, a, r) = 0$. Hence the supremum restricts to $r \in Q_2$. By the induction hypothesis:

$$\begin{aligned} (\eta_1 + \eta_2)^*(q, av, p) &= \bigvee_{r \in Q_2} \{(\eta_1 + \eta_2)(q, a, r) \wedge (\eta_1 + \eta_2)^*(r, v, p)\} \\ &= \bigvee_{r \in Q_2} \{\eta_2(q, a, r) \wedge \eta_2^*(r, v, p)\} \\ &= \eta_2^*(q, av, p). \end{aligned}$$

- (3) Let $q \in Q_1$ and $p \in Q_2$ (or vice versa). For any intermediate state $r \in Q_1 \cup Q_2$:

- If $r \in Q_1$, then by the induction hypothesis $(\eta_1 + \eta_2)^*(r, v, p) = 0$ because $r \in Q_1$ and $p \in Q_2$.
- If $r \in Q_2$, then $(\eta_1 + \eta_2)(q, a, r) = 0$ because $q \in Q_1$ and $r \in Q_2$.

In either case, $(\eta_1 + \eta_2)(q, a, r) \wedge (\eta_1 + \eta_2)^*(r, v, p) = 0$ for all $r \in Q_1 \cup Q_2$. Therefore,

$$(\eta_1 + \eta_2)^*(q, av, p) = 0.$$

Thus, by mathematical induction, the result holds for all $w \in X^*$.

Theorem 3.12. Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be Γ_k -reset fuzzy automata with $Q_1 \cap Q_2 = \emptyset$. Then the sum $F_1 + F_2$ is a componentwise Γ -reset fuzzy automaton with respect to $\Gamma = \Gamma_1 \wedge \Gamma_2$.

Proof. Let $F_k = (Q_k, X, \eta_k)$, $k = 1, 2$ be Γ_k -reset fuzzy automata with $Q_1 \cap Q_2 = \emptyset$. The sum of F_1 and F_2 is given by

$$F_1 + F_2 = (Q_1 \cup Q_2, X, \eta_1 + \eta_2).$$

Since F_1 and F_2 are Γ_1 -reset and Γ_2 -reset fuzzy automata respectively, there exist reset words $w_1, w_2 \in X^*$ and states $p_1 \in Q_1$, $p_2 \in Q_2$ such that

$$\eta_1^*(q_1, w_1, p_1) \geq \Gamma_1 \quad \text{for all } q_1 \in Q_1,$$

and

$$\eta_2^*(q_2, w_2, p_2) \geq \Gamma_2 \quad \text{for all } q_2 \in Q_2.$$

By the Sum Lemma, for any $q, p \in Q_1 \cup Q_2$ and $w \in X^*$, we have:

$$(\eta_1 + \eta_2)^*(q, w, p) = \begin{cases} \eta_1^*(q, w, p) & \text{if } q, p \in Q_1, \text{ and } w \in X_1, \\ \eta_2^*(q, w, p) & \text{if } q, p \in Q_2, \text{ and } w \in X_2, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, for the reset words w_1 and w_2 :

(1) For every $q \in Q_1$, choosing target state $p_1 \in Q_1$:

$$(\eta_1 + \eta_2)^*(q, w_1, p_1) = \eta_1^*(q, w_1, p_1) \geq \Gamma_1 \geq \Gamma_1 \wedge \Gamma_2 = \Gamma.$$

(2) For every $q \in Q_2$, choosing target state $p_2 \in Q_2$:

$$(\eta_1 + \eta_2)^*(q, w_2, p_2) = \eta_2^*(q, w_2, p_2) \geq \Gamma_2 \geq \Gamma_1 \wedge \Gamma_2 = \Gamma.$$

Thus, each component Q_k in $F_1 + F_2$ is reset to $p_k \in Q_k$ with degree at least $\Gamma = \Gamma_1 \wedge \Gamma_2 > 0$. Hence, $F_1 + F_2$ is a componentwise Γ -reset fuzzy automaton.

3.5. Cartesian Composition.

Definition 3.13. [2] Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata with $X_1 \cap X_2 = \emptyset$. The **Cartesian composition** of F_1 and F_2 is denoted by $F_1 \bullet F_2 = (Q_1 \times Q_2, X_1 \cup X_2, \eta_1 \bullet \eta_2)$, where the fuzzy transition relation

$$(\eta_1 \bullet \eta_2) : (Q_1 \times Q_2) \times (X_1 \cup X_2) \times (Q_1 \times Q_2) \rightarrow [0, 1]$$

is defined by

$$(\eta_1 \bullet \eta_2)((q_1, q_2), a, (p_1, p_2)) = \begin{cases} \eta_1(q_1, a, p_1) & \text{if } a \in X_1 \text{ and } q_2 = p_2, \\ \eta_2(q_2, a, p_2) & \text{if } a \in X_2 \text{ and } q_1 = p_1, \\ 0 & \text{otherwise} \end{cases}$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $a \in X_1 \cup X_2$.

Lemma 3.14. Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be fuzzy automata with $X_1 \cap X_2 = \emptyset$, and let $F_1 \bullet F_2$ be the Cartesian composition of F_1 and F_2 . Then for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $w \in (X_1 \cup X_2)^*$

$$(\eta_1 \bullet \eta_2)^*((q_1, q_2), w, (p_1, p_2)) = \begin{cases} \eta_1(q_1, w, p_1) & \text{if } w \in X_1^* \text{ and } q_2 = p_2, \\ \eta_2(q_2, w, p_2) & \text{if } w \in X_2^* \text{ and } q_1 = p_1, \\ 0 & \text{otherwise} \end{cases}$$

Proof. The proof follows by induction on $|w| = n$.

We prove Part (1) by induction on $|w| = n$ for $w \in X_1^*$. (Part (2) follows analogously for $w \in X_2^*$).

Case-I: Let $n = 1$ and $w = a \in X_1$. Since $X_1 \cap X_2 = \emptyset$, $a \notin X_2$. By Definition(3.13)

$$\begin{aligned} (\eta_1 \bullet \eta_2)^*((q_1, q_2), a, (p_1, p_2)) &= (\eta_1 \bullet \eta_2)((q_1, q_2), a, (p_1, p_2)) \\ &= \begin{cases} \eta_1(q_1, a, p_1) & \text{if } q_2 = p_2, \\ 0 & \text{if } q_2 \neq p_2. \end{cases} \\ &= \begin{cases} \eta_1^*(q_1, a, p_1) & \text{if } q_2 = p_2, \\ 0 & \text{if } q_2 \neq p_2. \end{cases} \end{aligned}$$

Thus, the result holds for $n = 1$.

Case-II: Assume that the result holds for all words in X_1^* of length $n - 1$. Let $w = av \in X_1^*$, where $a \in X_1$, $v \in X_1^*$ with $|v| = n - 1$. By the definition of the extended transition function:

$$\begin{aligned} (\eta_1 \bullet \eta_2)^*((q_1, q_2), av, (p_1, p_2)) &= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} \left\{ (\eta_1 \bullet \eta_2)((q_1, q_2), a, (r_1, r_2)) \right. \\ &\quad \left. \wedge (\eta_1 \bullet \eta_2)^*((r_1, r_2), v, (p_1, p_2)) \right\}. \end{aligned}$$

Since $a \in X_1$, $(\eta_1 \bullet \eta_2)((q_1, q_2), a, (r_1, r_2)) \neq 0$ only if $r_2 = q_2$. Thus, the supremum over $r_2 \in Q_2$ collapses to $r_2 = q_2$:

$$(\eta_1 \bullet \eta_2)^*((q_1, q_2), av, (p_1, p_2)) = \bigvee_{r_1 \in Q_1} \{ \eta_1(q_1, a, r_1) \wedge (\eta_1 \bullet \eta_2)^*((r_1, q_2), v, (p_1, p_2)) \}.$$

We now consider the following two cases:

- (1) **Let $q_2 = p_2$:** By the induction hypothesis for $v \in X_1^*$, since $r_2 = q_2 = p_2$, we have $(\eta_1 \bullet \eta_2)^*((r_1, q_2), v, (p_1, p_2)) = \eta_1^*(r_1, v, p_1)$. Therefore:

$$\begin{aligned} (\eta_1 \bullet \eta_2)^*((q_1, q_2), av, (p_1, p_2)) &= \bigvee_{r_1 \in Q_1} \{ \eta_1(q_1, a, r_1) \wedge \eta_1^*(r_1, v, p_1) \} \\ &= \eta_1^*(q_1, av, p_1). \end{aligned}$$

- (2) **Let $q_2 \neq p_2$:** Since $r_2 = q_2 \neq p_2$, by the induction hypothesis $(\eta_1 \bullet \eta_2)^*((r_1, q_2), v, (p_1, p_2)) = 0$ for all $r_1 \in Q_1$. Thus:

$$(\eta_1 \bullet \eta_2)^*((q_1, q_2), av, (p_1, p_2)) = 0.$$

Thus, by mathematical induction, Part (1) holds for all $w \in X_1^*$. By symmetric arguments, Part (2) holds for all $w \in X_2^*$.

Theorem 3.15. Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be Γ_k -reset fuzzy automata with $X_1 \cap X_2 = \emptyset$. Then the Cartesian composition $F_1 \bullet F_2$ is a Γ -reset fuzzy automaton with respect to $\Gamma = \Gamma_1 \wedge \Gamma_2$.

Proof. Let $F_k = (Q_k, X_k, \eta_k)$, $k = 1, 2$ be Γ_k -reset fuzzy automata with $X_1 \cap X_2 = \emptyset$. The Cartesian composition of F_1 and F_2 is given by

$$F_1 \bullet F_2 = (Q_1 \times Q_2, X_1 \cup X_2, \eta_1 \bullet \eta_2).$$

Since F_1 is a Γ_1 -reset fuzzy automaton and F_2 is a Γ_2 -reset fuzzy automaton, there exist reset words $w_1 \in X_1^*$, $w_2 \in X_2^*$ and states $p_1 \in Q_1$, $p_2 \in Q_2$ such that

$$\eta_1^*(q_1, w_1, p_1) \geq \Gamma_1 \quad \text{for all } q_1 \in Q_1,$$

and

$$\eta_2^*(q_2, w_2, p_2) \geq \Gamma_2 \quad \text{for all } q_2 \in Q_2.$$

Consider the concatenated word $w = w_1 w_2 \in (X_1 \cup X_2)^*$. For any arbitrary initial state $(q_1, q_2) \in Q_1 \times Q_2$ and target state $(p_1, p_2) \in Q_1 \times Q_2$:

$$\begin{aligned} (\eta_1 \bullet \eta_2)^*((q_1, q_2), w_1 w_2, (p_1, p_2)) &= \bigvee_{(r_1, r_2) \in Q_1 \times Q_2} \left\{ (\eta_1 \bullet \eta_2)^*((q_1, q_2), w_1, (r_1, r_2)) \right. \\ &\quad \left. \wedge (\eta_1 \bullet \eta_2)^*((r_1, r_2), w_2, (p_1, p_2)) \right\}. \end{aligned}$$

By Lemma (3.14), since $w_1 \in X_1^*$, the term $(\eta_1 \bullet \eta_2)^*((q_1, q_2), w_1, (r_1, r_2))$ is non-zero only when $r_2 = q_2$, in which case it equals $\eta_1^*(q_1, w_1, r_1)$. Similarly, since $w_2 \in X_2^*$, the term $(\eta_1 \bullet \eta_2)^*((r_1, r_2), w_2, (p_1, p_2))$ is non-zero only when $r_1 = p_1$, in which case it equals $\eta_2^*(r_2, w_2, p_2)$.

Choosing the specific intermediate state $(r_1, r_2) = (p_1, q_2) \in Q_1 \times Q_2$, we obtain:

$$\begin{aligned} (\eta_1 \bullet \eta_2)^*((q_1, q_2), w_1 w_2, (p_1, p_2)) &\geq (\eta_1 \bullet \eta_2)^*((q_1, q_2), w_1, (p_1, q_2)) \wedge (\eta_1 \bullet \eta_2)^*((p_1, q_2), w_2, (p_1, p_2)) \\ &= \eta_1^*(q_1, w_1, p_1) \wedge \eta_2^*(q_2, w_2, p_2) \\ &\geq \Gamma_1 \wedge \Gamma_2 = \Gamma. \end{aligned}$$

Since this inequality holds for every $(q_1, q_2) \in Q_1 \times Q_2$, the composite word $w = w_1 w_2 \in (X_1 \cup X_2)^*$ resets the entire state set $Q_1 \times Q_2$ to the target state (p_1, p_2) with membership value at least $\Gamma = \Gamma_1 \wedge \Gamma_2 > 0$.

Hence, $F_1 \bullet F_2$ is a Γ -reset fuzzy automaton.

4. CONCLUSION

In this paper, we studied the properties of Γ -reset fuzzy automaton. We proved that the cascade product, restricted cascade product, and wreath product sum and Cartesian composition of Γ -reset fuzzy automata are Γ -reset fuzzy automaton. These results extend the existing results on direct product and restricted direct product to more general product constructions in fuzzy automata theory. Future work may include investigating other product constructions and exploring the minimality and reducibility of such products.

REFERENCES

- [1] V. Karthikeyan, "Γ-Generalized directable fuzzy automata," *Journal of Applied Mathematics and Computational Mechanics*, 2018.
- [2] H. V. Kumbhojkar, & S. R. Chaudhari, "On covering of products of fuzzy finite state machines," *Fuzzy Sets and Systems*, 2002, 125(2), 215-222.
- [3] D. S. Malik, John N. Mordeson, and Mrinal K. Sen, *Products of fuzzy finite state machines*, Fuzzy Sets and Systems, 1997, 92(1), 95-102.
- [4] N. Mohanarao and V. Karthikeyan, "Products of Γ -reset fuzzy automata," *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, vol. 12, no. 11, pp. 6864-6870, 2021.
- [5] W. G. Wee, *On generalizations of adaptive algorithms and application of the fuzzy sets concept to pattern classification*, Ph.D. dissertation, Purdue University, West Lafayette, IN, USA, 1967.
- [6] W. G. Wee and K. S. Fu, "A formulation of fuzzy automata and its application as a model of learning systems," *IEEE Transactions on Systems Science and Cybernetics*, vol. 5, no. 3, pp. 215-223, 1969.



- [7] Young Bae Jun, *Intuitionistic fuzzy finite switchboard state machines*, Journal of Applied Mathematics and Computing, May 2006, 20(1-2),315-325
- [8] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965.