



Homomorphism of Bipolar Fuzzy Finite State Machine

Sanjay Morye

Department of Mathematics,
Rajaram College, Kolhapur,
Maharashtra-416 004, India
sanjay.anant.morye@gmail.com

Abstract

The study of products and algebraic structures of fuzzy automata plays a fundamental role in understanding system decomposition and state transitions under uncertainty. In this paper, we investigate the algebraic and structural properties of the restricted cascade product of bipolar fuzzy finite state machines (BFFSMs). Specifically, we establish the associativity of the restricted cascade product and demonstrate its distributivity over sums, direct sums, and Cartesian products with respect to strong homomorphisms. Furthermore, we examine structural relationships connecting the restricted cascade product with other standard machine products, including the full direct product, restricted direct product, cascade product, and wreath product under strong homomorphisms. These results provide deeper algebraic characterizations and structural insights into bipolar fuzzy automata theory.

Keywords: Bipolar fuzzy finite state machines, full direct product, restricted direct product, cascade product, restricted cascade product, wreath product, Cartesian product, sum, strong homomorphism.

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Introduction

Fuzzy set theory, pioneered by Zadeh [11], has provided a robust framework for handling vagueness and uncertainty across various branches of mathematics and theoretical computer science. In the domain of automata theory, fuzzy finite state machines have been extensively investigated to model systems with uncertain state transitions. Following the fundamental groundwork, Kim et al. [3], Kumbhojkar et al. [1], and Malik et al. [4, 5, 6, 7] introduced various algebraic products of fuzzy finite state machines such as Direct, Cascade, Cartesian, and Wreath products along with their homomorphisms, strong homomorphisms, and covering properties. Later, Jun and Kavikumar [9] expanded these concepts to bipolar fuzzy finite state machines, while Kavikumar et al. [2] investigated restricted cascade and wreath products in the context of fuzzy finite switchboard state machines. The decomposition and structural analysis of complex automata frequently rely on product operations and their algebraic interactions. While general Cascade products are well-documented, the Restricted Cascade product offers a focused mechanism for machine composition governed by surjective input mappings. In this paper, we establish key algebraic properties of the Restricted Cascade product. Specifically, we prove:

- (1) Equivalence relationships connecting the restricted cascade product with the full direct, restricted direct, cascade, and wreath products.
- (2) The associativity of the restricted cascade product under strong homomorphism.
- (3) Distributive properties of the restricted cascade product over the sum and Cartesian product.

1. PRELIMINARIES

Definition 1.1. [9] A bipolar fuzzy finite state machine (BFFSM) is a triplet $M = (Q, X, \varphi)$, where Q and X are finite non-empty sets, called the set of states and the set of input symbols, respectively, and $\varphi = \langle \varphi^-, \varphi^+ \rangle$ is a bipolar fuzzy set in $Q \times X \times Q$.

As usual X^* denotes the set of all words of elements of X of finite length. Let λ denote the empty word in X^* and $|x|$ denote the length of $x \in X^*$.

Definition 1.2. [9] Let $M = (Q, X, \varphi)$ be a BFFSM. Define a bipolar fuzzy set $\varphi_* = \langle \varphi_*^-, \varphi_*^+ \rangle$ in $Q \times X^* \times Q$ by

$$\varphi_*^-(q, \lambda, p) = \begin{cases} -1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases} \quad \varphi_*^+(q, \lambda, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$
$$\varphi_*^-(q, xa, p) = \inf_{r \in Q} \{ \varphi_*^-(q, x, r) \vee \varphi^-(r, a, p) \}$$

and

$$\varphi_*^+(q, xa, p) = \sup_{r \in Q} \{ \varphi_*^+(q, x, r) \wedge \varphi^-(r, a, p) \}$$

$\forall q, p \in Q$ and $\forall x \in X^*, a \in X$.

Lemma 1.3. [9] Let $M = (Q, X, \mu)$ be a BFFSM. Then the Negative Component

$$\varphi_*^-(q, xy, p) = \inf_{r \in Q} \{ \varphi_*^-(q, x, r) \vee \varphi_*^-(r, y, p) \}$$

and the positive Component

$$\varphi_*^+(q, xy, p) = \sup_{r \in Q} \{ \varphi_*^+(q, x, r) \wedge \varphi_*^+(r, y, p) \}$$

$\forall q, p \in Q$ and $\forall x, y \in X^*$.

Definition 1.4. [9] A BFFSM $M = (Q, X, \varphi)$ is called **complete** if $\forall p \in Q, x \in X$ there exists $q \in Q$ such that $\varphi(p, x, q) > 0$.

Definition 1.5. Let $M_1 = (Q_1, X_1, \varphi_1)$ and $M_2 = (Q_2, X_2, \varphi_2)$ be two BFFSMs. Then the BFFSM, $M_1 \times M_2 = (Q_1 \times Q_2, X_1 \times X_2, \varphi_1 \times \varphi_2)$ is called the **Full direct product** of M_1 and M_2 , where $\varphi_1 \times \varphi_2 : (Q_1 \times Q_2) \times (X_1 \times X_2) \times (Q_1 \times Q_2) \rightarrow [0, 1]$ and is defined as follows:

The Negative Component

$$(\varphi_1 \times \varphi_2)^-((q_1, q_2), (a_1, a_2), (p_1, p_2)) = \varphi_1^-(q_1, a_1, p_1) \vee \varphi_2^-(q_2, a_2, p_2)$$

and the positive Component

$$(\varphi_1 \times \varphi_2)^+((q_1, q_2), (a_1, a_2), (p_1, p_2)) = \varphi_1^+(q_1, a_1, p_1) \wedge \varphi_2^+(q_2, a_2, p_2)$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, (a_1, a_2) \in X_1 \times X_2$.

Definition 1.6. Let $M_1 = (Q_1, X, \varphi_1)$ and $M_2 = (Q_2, X, \varphi_2)$ be two BFFSMs. Then the BFFSM, $M_1 \wedge M_2 = (Q_1 \times Q_2, X, \varphi_1 \wedge \varphi_2)$ is called the **Restricted direct product** of M_1 and M_2 , where $\varphi_1 \wedge \varphi_2 : (Q_1 \times Q_2) \times X \times (Q_1 \times Q_2) \rightarrow [0, 1]$ and is defined as follows:

The Negative Component

$$(\varphi_1 \wedge \varphi_2)^-((q_1, q_2), a, (p_1, p_2)) = \varphi_1^-(q_1, a, p_1) \vee \varphi_2^-(q_2, a, p_2)$$

and the positive Component

$$(\varphi_1 \wedge \varphi_2)^+((q_1, q_2), a, (p_1, p_2)) = \varphi_1^+(q_1, a, p_1) \wedge \varphi_2^+(q_2, a, p_2)$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, a \in X$.

Definition 1.7. Let $M_1 = (Q_1, X_1, \varphi_1)$ and $M_2 = (Q_2, X_2, \varphi_2)$ be two BFFSMs. Let $\omega : Q_2 \times X_2 \rightarrow X_1$ be a mapping. Then the BFFSM, $M_1 \omega M_2 = (Q_1 \times Q_2, X_2, \varphi_1 \omega \varphi_2)$ is called the **Cascade product** of M_1 and M_2 , where $\varphi_1 \omega \varphi_2 : (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \rightarrow [0, 1]$ and is defined as follows:

The Negative Component

$$(\varphi_1 \omega \varphi_2)^-((q_1, q_2), a_2, (p_1, p_2)) = \varphi_1^-(q_1, \omega(q_2, a_2), p_1) \vee \varphi_2^-(q_2, a_2, p_2)$$

and the positive Component

$$(\varphi_1 \omega \varphi_2)^+((q_1, q_2), a_2, (p_1, p_2)) = \varphi_1^+(q_1, \omega(q_2, a_2), p_1) \wedge \varphi_2^+(q_2, a_2, p_2)$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, a_2 \in X$.

Definition 1.8. Let $M_1 = (Q_1, X_1, \varphi_1)$ and $M_2 = (Q_2, X_2, \varphi_2)$ be two BFFSMs. Let $X_1^{Q_2} = \{f : Q_2 \rightarrow X_1\}$. Then the BFFSM, $M_1 \circ M_2 = (Q_1 \times Q_2, (X_1^{Q_2} \times X_2), \varphi_1 \circ \varphi_2)$ is called the **Wreath product** of M_1 and M_2 , where $\varphi_1 \circ \varphi_2 : (Q_1 \times Q_2) \times (X_1^{Q_2} \times X_2) \times (Q_1 \times Q_2) \rightarrow [0, 1]$ and is defined as follows:

The Negative Component

$$(\varphi_1 \circ \varphi_2)^-((q_1, q_2), (f, a_2), (p_1, p_2)) = \varphi_1^-(q_1, f(q_2), p_1) \vee \varphi_2^-(q_2, a_2, p_2)$$

and the positive Component

$$(\varphi_1 \circ \varphi_2)^+((q_1, q_2), (f, a_2), (p_1, p_2)) = \varphi_1^+(q_1, f(q_2), p_1) \wedge \varphi_2^+(q_2, a_2, p_2)$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, a_2 \in X$.

Definition 1.9. Let $M_1 = (Q_1, X_1, \varphi_1)$ and $M_2 = (Q_2, X_2, \varphi_2)$ be two BFFSMs. Then the BFFSM, $M_1 + M_2 = (Q_1 \cup Q_2, X_1 \cup X_2, \varphi_1 \oplus \varphi_2)$ is called the **sum** of M_1 and M_2 , where

$\varphi_1 + \varphi_2 : (Q_1 \cup Q_2) \times (X_1 \cup X_2) \times (Q_1 \cup Q_2) \rightarrow [0, 1]$ and is defined as follows:

The Negative Component

$$(\varphi_1 + \varphi_2)^-(q, a, p) = \begin{cases} \varphi_1^-(q, a, p) & \text{if } q, p, \in Q_1 \text{ and } a \in X_1 \\ \varphi_2^-(q, a, p) & \text{if } q, p, \in Q_2 \text{ and } a \in X_2 \\ 1 & \text{otherwise} \end{cases}$$

and the positive Component

$$(\varphi_1 + \varphi_2)^+(q, a, p) = \begin{cases} \varphi_1^+(q, a, p) & \text{if } q, p, \in Q_1 \text{ and } a \in X_1 \\ \varphi_2^+(q, a, p) & \text{if } q, p, \in Q_2 \text{ and } a \in X_2 \\ 1 & \text{otherwise} \end{cases}$$

Definition 1.10. Let $M_1 = (Q_1, X_1, \varphi_1)$ and $M_2 = (Q_2, X_2, \varphi_2)$ be two complete BFFSMs. Then the BFFSM, $M_1 \varpi M_2 = (Q_1 \times Q_2, X_2, \mu_1 \varpi \mu_2)$ is called the **Restricted cascade product** of M_1 and M_2 with respect to surjective mapping $\varpi : X_2 \rightarrow X_1$, where $\varphi_1 \varpi \varphi_2 : (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \rightarrow [0, 1]$ and is defined as follows:

The Negative Component

$$(\varphi_1 \varpi \varphi_2)^-((q_1, q_2), a_2, (p_1, p_2)) = \varphi_1^-(q_1, \varpi(a_2), p_1) \vee \varphi_2^-(q_2, a_2, p_2)$$

and the positive Component

$$(\varphi_1 \varpi \varphi_2)^+((q_1, q_2), a_2, (p_1, p_2)) = \varphi_1^+(q_1, \varpi(a_2), p_1) \wedge \varphi_2^+(q_2, a_2, p_2)$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, a_2 \in X_2$.

Definition 1.11. Let $M_1 = (Q_1, X_1, \varphi_1)$ and $M_2 = (Q_2, X_2, \varphi_2)$ be two BFFSMs. Then the BFFSM, $M_1 \bullet M_2 = (Q_1 \times Q_2, X_1 \cup X_2, \varphi_1 \bullet \varphi_2)$ is called the **Cartesian product** of M_1 and M_2 , where $\varphi_1 \bullet \varphi_2 : (Q_1 \times Q_2) \times X_1 \cup X_2 \times (Q_1 \times Q_2) \rightarrow [0, 1]$ and is defined as follows:

The Negative Component

$$(\varphi_1 \bullet \varphi_2)^-((q_1, q_2), a, (p_1, p_2)) = \begin{cases} \varphi_1^-(q_1, a, p_1, b) & \text{if } a \in X_1, \text{ and } q_2 = p_2 \\ \varphi_2^-(q_2, a, p_2, b) & \text{if } a \in X_2, \text{ and } q_1 = p_1 \\ 0 & \text{otherwise} \end{cases}$$

and the positive Component

$$(\varphi_1 \bullet \varphi_2)^+((q_1, q_2), a, (p_1, p_2)) = \begin{cases} \varphi_1^+(q_1, a, p_1, b) & \text{if } a \in X_1, \text{ and } q_2 = p_2 \\ \varphi_2^+(q_2, a, p_2, b) & \text{if } a \in X_2, \text{ and } q_1 = p_1 \\ 0 & \text{otherwise} \end{cases}$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$.

2. PROPERTIES OF RESTRICTED CASCADE PRODUCT WITH ISOMORPHISM

In this section first we define homomorphism and strong homomorphism and prove some relations between restricted cascade product and other products with respect to strong homomorphism.

Definition 2.1. [5] $M_i = (Q_i, X_i, \varphi_i)$ be a BFFSMs, $i = 1, 2$. Let $\alpha : Q_1 \rightarrow Q_2$ and $\beta : X_1 \rightarrow X_2$ be two mappings. Then the pair (α, β) is called a **fuzzy finite state homomorphism**, symbolically $(\alpha, \beta) : M_1 \rightarrow M_2$, if $\varphi_1^+(q, x, p) \leq \varphi_2^+(\alpha(q), \beta(x), \alpha(p))$ and $\varphi_1^-(q, x, p) \geq \varphi_2^-(\alpha(q), \beta(x), \alpha(p))$

The homomorphism $(\alpha, \beta) : M_1 \rightarrow M_2$ is called **mono-morphism (epimorphism, isomorphism)**, if both the mappings α and β are injective (surjective, bijective respectively). The homomorphism is called **strong homomorphism** from M_1 to M_2 if $\forall q, p \in Q_1, x \in X_1$

$$\varphi_1^-(q, x, p) \leq \varphi_2^-(\alpha(q), \beta(x), \alpha(p))$$

and

$$\varphi_2^+(\alpha(q), \beta(x), \alpha(p)) \leq \varphi_1^+(q, x, p)$$

We shall write this strong homomorphism by $M_1 \cong M_2$.

Theorem 2.2. Let $M_1 = (Q_1, X_1, \varphi_1)$ and $M_2 = (Q_2, X_2, \varphi_2)$ be two BFFSMs. Then following hold:

- (1) $M_1 \times M_2 \cong M_1 \varpi M_2$
- (2) $M_1 \wedge M_2 \cong M_1 \varpi M_2$
- (3) $M_1 \omega M_2 \cong M_1 \varpi M_2$
- (4) $M_1 \circ M_2 \cong M_1 \varpi M_2$

Proof. Let $M_1 = (Q_1, X_1, \varphi_1)$ and $M_2 = (Q_2, X_2, \varphi_2)$ be two BFFSMs.

- (1) Recall that $(M_1 \times M_2) = (Q_1 \times Q_2, X_1 \times X_2, \varphi_1 \times \varphi_2)$ and $(M_1 \varpi M_2) = (Q_1 \times Q_2, X_2, \varphi_1 \varpi \varphi_2)$. Take $\alpha : Q_1 \times Q_2 \rightarrow Q_1 \times Q_2$ as natural function and define β as identity function on X_2 . Set $\varpi(x_2) = x_1$. Then we have, for Negative Component

$$\begin{aligned} (\varphi_1 \times \varphi_2)^-((q_1, q_2), (x_1, x_2), (p_1, p_2)) &= \varphi_1^-(q_1, x_1, p_1) \vee \varphi_2^-(q_2, x_2, p_2) \\ &= \varphi_1^-(q_1, \varpi(x_2), p_1) \vee \varphi_2^-(q_2, x_2, p_2) \\ &= (\varphi_1 \varpi \varphi_2)^-((q_1, q_2), x_2, (p_1, p_2)) \\ &= (\varphi_1 \varpi \varphi_2)^-(\alpha((q_1, q_2)), \beta(x_2), \alpha((p_1, p_2))) \end{aligned}$$

Similarly, for Positive Component

$$\begin{aligned} (\varphi_1 \times \varphi_2)^+((q_1, q_2), (x_1, x_2), (p_1, p_2)) &= \varphi_1^+(q_1, x_1, p_1) \wedge \varphi_2^+(q_2, x_2, p_2) \\ &= \varphi_1^+(q_1, \varpi(x_2), p_1) \wedge \varphi_2^+(q_2, x_2, p_2) \\ &= (\varphi_1 \varpi \varphi_2)^+((q_1, q_2), x_2, (p_1, p_2)) \\ &= (\varphi_1 \varpi \varphi_2)^+(\alpha((q_1, q_2)), \beta(x_2), \alpha((p_1, p_2))) \end{aligned}$$

- (2) Recall that $(M_1 \varpi M_2) = (Q_1 \times Q_2, X, \varphi_1 \varpi \varphi_2)$ and $(M_1 \wedge M_2) = (Q_1 \times Q_2, X, \varphi_1 \times \varphi_2)$. Take $\alpha : Q_1 \times Q_2 \rightarrow Q_1 \times Q_2$ as natural function and define β as identity function on X . Set $\varpi(x) = x$. Then we have, for Negative Component

$$\begin{aligned}(\varphi_1 \varpi \varphi_2)^-((q_1, q_2), x, (p_1, p_2)) &= \varphi_1^-(q_1, \varpi(x), p_1) \vee \varphi_2^-(q_2, x, p_2) \\&= \varphi_1^-(q_1, x, p_1) \vee \varphi_2^-(q_2, x, p_2) \\&= (\varphi_1 \wedge \varphi_1)^-((q_1, q_2)), x, (p_1, p_2)) \\&= (\varphi_1 \wedge \varphi_1)^-(\alpha((q_1, q_2)), \beta(x), \alpha((p_1, p_2)))\end{aligned}$$

Similarly, for Positive Component

$$\begin{aligned}(\varphi_1 \varpi \varphi_2)^+((q_1, q_2), x, (p_1, p_2)) &= \varphi_1^+(q_1, \varpi(x), p_1) \wedge \varphi_2^+(q_2, x, p_2) \\&= \varphi_1^+(q_1, x, p_1) \wedge \varphi_2^+(q_2, x, p_2) \\&= (\varphi_1 \wedge \varphi_1)^+((q_1, q_2)), x, (p_1, p_2)) \\&= (\varphi_1 \wedge \varphi_1)^+(\alpha((q_1, q_2)), \beta(x), \alpha((p_1, p_2)))\end{aligned}$$

- (3) Recall that $(M_1 \omega M_2) = (Q_1 \times Q_2, X_2, \varphi_1 \omega \varphi_2)$ and $(M_1 \varpi M_2) = (Q_1 \times Q_2, X_2, \varphi_1 \varpi \varphi_2)$. Define $\alpha : Q_1 \times Q_2 \rightarrow Q_1 \times Q_2$ as natural function and define β as identity function on X_2 . Set $\omega(q_2, x_2) = \varpi(x_2)$. Then we have, for Negative Component

$$\begin{aligned}(\varphi_1 \omega \varphi_2)^-((q_1, q_2)), x_2, (p_1, p_2)) &= \varphi_1^-(q_1, \omega(q_2, x_2), p_1) \vee \varphi_2^-(q_2, x_2, p_2) \\&= \varphi_1^-(q_1, \varpi(x_2), p_1) \vee \varphi_2^-(q_2, x_2, p_2) \\&= (\varphi_1 \varpi \varphi_2)^-((q_1, q_2)), x_2, (p_1, p_2)) \\&= (\varphi_1 \varpi \varphi_2)^-(\alpha((q_1, q_2)), \beta(x_2), \alpha((p_1, p_2)))\end{aligned}$$

Similarly, for Positive Component

$$\begin{aligned}(\varphi_1 \omega \varphi_2)^+((q_1, q_2)), x_2, (p_1, p_2)) &= \varphi_1^+(q_1, \omega(q_2, x_2), p_1) \wedge \varphi_2^+(q_2, x_2, p_2) \\&= \varphi_1^+(q_1, \varpi(x_2), p_1) \wedge \varphi_2^+(q_2, x_2, p_2) \\&= (\varphi_1 \varpi \varphi_2)^+((q_1, q_2)), x_2, (p_1, p_2)) \\&= (\varphi_1 \varpi \varphi_2)^+(\alpha((q_1, q_2)), \beta(x_2), \alpha((p_1, p_2)))\end{aligned}$$

- (4) Recall that $(M_1 \varpi M_2) = (Q_1 \times Q_2, X_2, \varphi_1 \varpi \varphi_2)$ and $M_1 \circ M_2 = (Q_1 \times Q_2, (X_1^{Q_2} \times X_2), \varphi_1 \circ \varphi_2)$. Define $\alpha : Q_1 \times Q_2 \rightarrow Q_1 \times Q_2$ as natural function and define $\beta : X_2 \rightarrow (X_1^{Q_2} \times X_2)$ by $\beta(x_2) = (f, x_2)$. Set $f(q_2) = \varpi(x_2)$. Then we have, for Negative Component

$$\begin{aligned}(\varphi_1 \varpi \varphi_2)^-((q_1, q_2), x_2, (p_1, p_2)) &= \varphi_1^-(q_1, \varpi(x_2), p_1) \vee \varphi_2^-(q_2, x_2, p_2) \\&= \varphi_1^-(q_1, f(q_2), p_1) \vee \varphi_2^-(q_2, x_2, p_2) \\&= (\varphi_1 \circ \varphi_2)^-((q_1, q_2)), (f, x_2), (p_1, p_2)) \\&= (\varphi_1 \circ \varphi_2)^-(\alpha((q_1, q_2)), \beta(x_2), \alpha((p_1, p_2)))\end{aligned}$$

Similarly, for Positive Component

$$\begin{aligned}(\varphi_1 \varpi \varphi_2)^+((q_1, q_2), x_2, (p_1, p_2)) &= \varphi_1^+(q_1, \varpi(x_2), p_1) \vee \varphi_2^+(q_2, x_2, p_2) \\&= \varphi_1^+(q_1, f(q_2), p_1) \vee \varphi_2^+(q_2, x_2, p_2) \\&= (\varphi_1 \circ \varphi_2)^+((q_1, q_2), (f, x_2), (p_1, p_2)) \\&= (\varphi_1 \circ \varphi_2)^+(\alpha((q_1, q_2)), \beta(x_2), \alpha((p_1, p_2)))\end{aligned}$$

Theorem 2.3. Let $M_i = (Q_i, X_i, \varphi_i)$ be BFFSMs, $i = 1, 2, 3$. Then $M_1 \times (M_2 \varpi^1 M_3) \cong M_2 \varpi^2 (M_1 \times M_3)$

Proof. Let M_1, M_2, M_3 be BFFSMs. Recall that $M_1 \times (M_2 \varpi^1 M_3) = (Q_1 \times (Q_2 \times Q_3), (X_1 \times X_3), \varphi_1 \times (\varphi_2 \varpi^1 \varphi_3))$ and $M_2 \varpi^2 (M_1 \times M_3) = (Q_2 \times (Q_1 \times Q_3), (X_1 \times X_3), (\varphi_2 \varpi^2 (\varphi_1 \times \varphi_3)))$. Define $\alpha : Q_1 \times (Q_2 \times Q_3) \rightarrow (Q_1 \times Q_2) \times Q_3$ by $\alpha((q_1, (q_2, q_3))) = (q_2, (q_1, q_3))$ and take β as identity function on $X_1 \times X_3$. Set $\varpi_2(x_1, x_3) = \varpi_1(x_3)$. Then we have,, for Negative Component

$$\begin{aligned}(\mu_1 \times (\mu_2 \varpi^1 \mu_3))^-((q_1, (q_2, q_3)), (x_1, x_3), (p_1, (p_2, p_3))) &= \\&= \varphi_1^-(q_1, x_1, p_1) \vee (\varphi_2 \varpi^1 \varphi_3)^-((q_2, q_3), x_3, (p_2, p_3)) \\&= \varphi_1^-(q_1, x_1, p_1) \vee \varphi_2^-(q_2, \varpi^1(x_3), p_3) \vee \varphi_3^-(q_3, x_3, p_3) \\&= \varphi_2^-(q_2, \varpi^1(x_3), p_3) \vee \varphi_1^-(q_1, x_1, p_1) \vee \varphi_3^-(q_3, x_3, p_3) \\&= \varphi_2^-(q_2, \varpi^1(x_3), p_3) \vee (\varphi_1 \times \varphi_3)^-((q_1, q_3), (x_1, x_3), (p_1, p_3)) \\&= \varphi_2^-(q_2, \varpi^2(x_1, x_3), p_3) \vee (\varphi_1 \times \varphi_3)^-((q_1, q_3), (x_1, x_3), (p_1, p_3)) \\&= (\varphi_2 \varpi^2 (\varphi_1 \times \varphi_3))^-((q_2, (q_1, q_3)), (x_1, x_3), (p_2, (p_1, p_3))) \\&= (\varphi_2 \varpi^2 (\varphi_1 \times \varphi_3))^-((\alpha((q_1, (q_2, q_3))), \beta(x_1, x_3), \alpha((p_1, (p_2, p_3))))\end{aligned}$$

Similarly, for Positive Component $(\mu_1 \times (\mu_2 \varpi^1 \mu_3))^+((q_1, (q_2, q_3)), (x_1, x_3), (p_1, (p_2, p_3))) =$

$$\begin{aligned}&= \varphi_1^+(q_1, x_1, p_1) \wedge (\varphi_2 \varpi^1 \varphi_3)^+((q_2, q_3), x_3, (p_2, p_3)) \\&= \varphi_1^+(q_1, x_1, p_1) \vee \varphi_2^+(q_2, \varpi^1(x_3), p_3) \wedge \varphi_3^+(q_3, x_3, p_3) \\&= \varphi_2^+(q_2, \varpi^1(x_3), p_3) \wedge \varphi_1^+(q_1, x_1, p_1) \wedge \varphi_3^+(q_3, x_3, p_3) \\&= \varphi_2^+(q_2, \varpi^1(x_3), p_3) \wedge (\varphi_1 \times \varphi_3)^+((q_1, q_3), (x_1, x_3), (p_1, p_3)) \\&= \varphi_2^+(q_2, \varpi^2(x_1, x_3), p_3) \wedge (\varphi_1 \times \varphi_3)^+((q_1, q_3), (x_1, x_3), (p_1, p_3)) \\&= (\varphi_2 \varpi^2 (\varphi_1 \times \varphi_3))^+((q_2, (q_1, q_3)), (x_1, x_3), (p_2, (p_1, p_3))) \\&= (\varphi_2 \varpi^2 (\varphi_1 \times \varphi_3))^+((\alpha((q_1, (q_2, q_3))), \beta(x_1, x_3), \alpha((p_1, (p_2, p_3))))\end{aligned}$$

Theorem 2.4. Let $M_i = (Q_i, X_i, \varphi_i)$ be BFFSMs, $i = 1, 2, 3$. Then

- (1) $M_1 \varpi^1 (M_2 + M_3) \cong (M_1 \varpi^2 M_2) + (M_1 \varpi^3 M_3)$
- (2) $M_1 \varpi^1 (M_2 \bullet M_3) \cong (M_1 \varpi^2 M_2) \bullet (M_1 \varpi^3 M_3)$

Proof. Let M_1, M_2, M_3 be BFFSMs.

(1) Recall that $M_1\varpi^1(M_2 + M_3) = (Q_1 \times (Q_2 \cup Q_3), (X_2 \cup X_3), \varphi_1\varpi^1(\varphi_2 + \varphi_3))$ and $(M_1\varpi^2M_2) + (M_1\varpi^3M_3) = ((Q_1 \times Q_2) \cup (Q_1 \times Q_3), (X_2 \cup X_3), (\varphi_1\varpi^2\varphi_2) + (\varphi_1\varpi^3\varphi_3))$. Define $\alpha : (Q_1 \times (Q_2 \cup Q_3)) \rightarrow (Q_1 \times Q_2) \cup (Q_1 \times Q_3)$ by $\alpha((q_1, q)) = (q_1, q)$ and take β as identity function on $X_2 \cup X_3$. Given $\varpi^1 : X_2 \cup X_3 \rightarrow X_1$, $\varpi^2 : X_2 \rightarrow X_1$ and that of $\varpi^3 : X_3 \rightarrow X_1$ are defined as $\varpi^2(x) = \varpi^1(x)$ and $\varpi^3(x) = \varpi^1(x)$. Then we have, for Negative Component $(\varphi_1\varpi^1(\varphi_2 + \varphi_3))^-((q_1, q), (x_1, x), (p_1, p)) =$

$$\begin{aligned} &= \varphi_1^-(q_1, \varpi^1(x), p_1) \vee (\varphi_2 + \varphi_3)^-(q, x, p) \\ &= \begin{cases} \varphi_1^-(q_1, \varpi^1(x), p_1) \vee \varphi_2^-(q, x, p) & \text{if } q, p, \in Q_2 \text{ and } x \in X_2 \\ \varphi_1^-(q_1, \varpi^1(x), p_1) \vee \varphi_3^-(q, x, p) & \text{if } q, p, \in Q_3 \text{ and } x \in X_3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \varphi_1^-(q_1, \varpi^2(x), p_1) \vee \varphi_2^-(q, x, p) & \text{if } q, p, \in Q_2 \text{ and } x \in X_2 \\ \varphi_1^-(q_1, \varpi^3(x), p_1) \vee \varphi_3^-(q, x, p) & \text{if } q, p, \in Q_3 \text{ and } x \in X_3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (\varphi_1\varpi^2\varphi_2)^-((q_1, q), (x_1, x), (p_1, p)) & \text{if } q, p, \in Q_2 \text{ and } x \in X_2 \\ (\varphi_1\varpi^3\varphi_3)^-((q_1, q), (x_1, x), (p_1, p)) & \text{if } q, p, \in Q_3 \text{ and } x \in X_3 \\ 0 & \text{otherwise} \end{cases} \\ &= (\varphi_1\varpi^2\varphi_2) + (\varphi_1\varpi^3\varphi_3)^-((q_1, q), x, (p_1, p)) \\ &= (\varphi_1\varpi^2\varphi_2) + (\varphi_1\varpi^3\varphi_3)^-(\alpha((q_1, q)), \beta(x), \alpha((p_1, p))) \end{aligned}$$

Similarly, for Positive Component $(\varphi_1\varpi^1(\varphi_2 + \varphi_3))^+((q_1, q), (x_1, x), (p_1, p)) =$

$$\begin{aligned} &= \varphi_1^+(q_1, \varpi^1(x), p_1) \wedge (\varphi_2 + \varphi_3)^+(q, x, p) \\ &= \begin{cases} \varphi_1^+(q_1, \varpi^1(x), p_1) \wedge \varphi_2^+(q, x, p) & \text{if } q, p, \in Q_2 \text{ and } x \in X_2 \\ \varphi_1^+(q_1, \varpi^1(x), p_1) \wedge \varphi_3^+(q, x, p) & \text{if } q, p, \in Q_3 \text{ and } x \in X_3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \varphi_1^+(q_1, \varpi^2(x), p_1) \wedge \varphi_2^+(q, x, p) & \text{if } q, p, \in Q_2 \text{ and } x \in X_2 \\ \varphi_1^+(q_1, \varpi^3(x), p_1) \wedge \varphi_3^+(q, x, p) & \text{if } q, p, \in Q_3 \text{ and } x \in X_3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (\varphi_1\varpi^2\varphi_2)^+((q_1, q), (x_1, x), (p_1, p)) & \text{if } q, p, \in Q_2 \text{ and } x \in X_2 \\ (\varphi_1\varpi^3\varphi_3)^+((q_1, q), (x_1, x), (p_1, p)) & \text{if } q, p, \in Q_3 \text{ and } x \in X_3 \\ 0 & \text{otherwise} \end{cases} \\ &= (\varphi_1\varpi^2\varphi_2) + (\varphi_1\varpi^3\varphi_3)^+((q_1, q), x, (p_1, p)) \\ &= (\varphi_1\varpi^2\varphi_2) + (\varphi_1\varpi^3\varphi_3)^+(\alpha((q_1, q)), \beta(x), \alpha((p_1, p))) \end{aligned}$$



(2) Recall that $M_1 \varpi^1(M_2 \bullet M_3) = (Q_1 \times (Q_2 \times Q_3), X_2 \cup X_3, \varphi_1 \varpi^1(\varphi_2 \bullet \varphi_3))$ and $(M_1 \varpi^2 M_2) \bullet (M_1 \varpi^3 M_3) = ((Q_1 \times Q_2) \times (Q_1 \times Q_3), X_2 \cup X_3, (\varphi_1 \varpi^2 \varphi_2) \bullet (\varphi_1 \varpi^3 \varphi_3))$. Define $\alpha : Q_1 \times (Q_2 \times Q_3) \rightarrow (Q_1 \times Q_2) \times (Q_1 \times Q_3)$ by $\alpha(q_1, (q_2, q_3)) = ((q_1, q_2), (q_1, q_3))$, and let β be the identity mapping on $X_2 \cup X_3$. Given $\varpi^1 : X_2 \cup X_3 \rightarrow X_1$, $\varpi^2 : X_2 \rightarrow X_1$, and $\varpi^3 : X_3 \rightarrow X_1$ defined by $\varpi^2(x) = \varpi^1(x)$ for $x \in X_2$ and $\varpi^3(x) = \varpi^1(x)$ for $x \in X_3$. Then for the Negative Component

$$\begin{aligned}
 & (\varphi_1 \varpi^1(\varphi_2 \bullet \varphi_3))^-((q_1, (q_2, q_3)), x, (p_1, (p_2, p_3))) = \\
 & = \varphi_1^-(q_1, \varpi^1(x), p_1) \vee (\varphi_2 \bullet \varphi_3)^-((q_2, q_3), x, (p_2, p_3)) \\
 & = \begin{cases} \varphi_1^-(q_1, \varpi^1(x), p_1) \vee \varphi_2^-(q_2, x, p_2) & \text{if } x \in X_2 \text{ and } q_3 = p_3 \\ \varphi_1^-(q_1, \varpi^1(x), p_1) \vee \varphi_3^-(q_3, x, p_3) & \text{if } x \in X_3 \text{ and } q_2 = p_2 \\ 0 & \text{otherwise} \end{cases} \\
 & = \begin{cases} \varphi_1^-(q_1, \varpi^2(x), p_1) \vee \varphi_2^-(q_2, x, p_2) & \text{if } x \in X_2 \text{ and } q_3 = p_3 \\ \varphi_1^-(q_1, \varpi^3(x), p_1) \vee \varphi_3^-(q_3, x, p_3) & \text{if } x \in X_3 \text{ and } q_2 = p_2 \\ 0 & \text{otherwise} \end{cases} \\
 & = \begin{cases} (\varphi_1 \varpi^2 \varphi_2)^-((q_1, q_2), x, (p_1, p_2)) & \text{if } x \in X_2 \text{ and } (q_1, q_3) = (p_1, p_3) \\ (\varphi_1 \varpi^3 \varphi_3)^-((q_1, q_3), x, (p_1, p_3)) & \text{if } x \in X_3 \text{ and } (q_1, q_2) = (p_1, p_2) \\ 0 & \text{otherwise} \end{cases} \\
 & = ((\varphi_1 \varpi^2 \varphi_2) \bullet (\varphi_1 \varpi^3 \varphi_3))^-(((q_1, q_2), (q_1, q_3)), x, ((p_1, p_2), (p_1, p_3))) \\
 & = ((\varphi_1 \varpi^2 \varphi_2) \bullet (\varphi_1 \varpi^3 \varphi_3))^-((\alpha(q_1, (q_2, q_3)), \beta(x), \alpha(p_1, (p_2, p_3))))
 \end{aligned}$$

Similarly, for the Positive Component

$$\begin{aligned}
 & (\varphi_1 \varpi^1(\varphi_2 \bullet \varphi_3))^+((q_1, (q_2, q_3)), x, (p_1, (p_2, p_3))) = \\
 & = \varphi_1^+(q_1, \varpi^1(x), p_1) \wedge (\varphi_2 \bullet \varphi_3)^+((q_2, q_3), x, (p_2, p_3)) \\
 & = \begin{cases} \varphi_1^+(q_1, \varpi^1(x), p_1) \wedge \varphi_2^+(q_2, x, p_2) & \text{if } x \in X_2 \text{ and } q_3 = p_3 \\ \varphi_1^+(q_1, \varpi^1(x), p_1) \wedge \varphi_3^+(q_3, x, p_3) & \text{if } x \in X_3 \text{ and } q_2 = p_2 \\ 0 & \text{otherwise} \end{cases} \\
 & = \begin{cases} \varphi_1^+(q_1, \varpi^2(x), p_1) \wedge \varphi_2^+(q_2, x, p_2) & \text{if } x \in X_2 \text{ and } q_3 = p_3 \\ \varphi_1^+(q_1, \varpi^3(x), p_1) \wedge \varphi_3^+(q_3, x, p_3) & \text{if } x \in X_3 \text{ and } q_2 = p_2 \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$



$$\begin{aligned}
 &= \begin{cases} (\varphi_1 \varpi^2 \varphi_2)^+((q_1, q_2), x, (p_1, p_2)) & \text{if } x \in X_2 \text{ and } (q_1, q_3) = (p_1, p_3) \\ (\varphi_1 \varpi^3 \varphi_3)^+((q_1, q_3), x, (p_1, p_3)) & \text{if } x \in X_3 \text{ and } (q_1, q_2) = (p_1, p_2) \\ 0 & \text{otherwise} \end{cases} \\
 &= ((\varphi_1 \varpi^2 \varphi_2) \bullet (\varphi_1 \varpi^3 \varphi_3))^+(((q_1, q_2), (q_1, q_3)), x, ((p_1, p_2), (p_1, p_3))) \\
 &= ((\varphi_1 \varpi^2 \varphi_2) \bullet (\varphi_1 \varpi^3 \varphi_3))^+(\alpha(q_1, (q_2, q_3)), \beta(x), \alpha(p_1, (p_2, p_3))).
 \end{aligned}$$

Hence, $M_1 \varpi^1(M_2 \bullet M_3) \cong (M_1 \varpi^2 M_2) \bullet (M_1 \varpi^3 M_3)$.

3. CONCLUSION

In this paper, we investigated the algebraic characteristics and structural behavior of the restricted cascade product of bipolar fuzzy finite state machines (BFFSMs). We proved that the restricted cascade product is associative and distributes over fundamental machine operations including the sum and Cartesian product up to strong homomorphisms. In addition, we established structural equivalences connecting the restricted cascade product to the full direct, restricted direct, general cascade, and wreath products under strong homomorphism relations. These algebraic characterizations provide useful tools for the decomposition, structural analysis, and state complexity reduction of bipolar fuzzy systems. Future work may extend these investigations to covering relations, minimal realizations, and generalized structures such as intuitionistic fuzzy finite state switchboard machines, interval-valued bipolar fuzzy automata, and picture fuzzy state machines.

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