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Sicherman Dice and Unique Factorization: An Overview

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Introduction:

Dice theory is about calculating basic probabilities which thus let you estimate the success and fail rates of model performance, equipment and abilities. Many different types of games involve the use of numbered dice. The most common type of die used is the standard six-sided die with the numbers one through six used on its six sides. What if there was another way of numbering a set of dice using only positive integers that would create the same probability outcomes as those of a standard set of dice? George Sicherman discovered an interesting pair of dice whose sums have the same probability distribution as a pair of standard dice. This pair of dice is unique. To prove the uniqueness of his combination the method, factorization of polynomials is used. We will see how unique factorization can be used to determine probabilities of rolls of dice. Are there any other labelling of two six-sided dice that will give the same outcomes as two regular dice?

First we see the simple situation involving two two-sided dice. Suppose we have a pair of two-sided dice, each with the numbers 1 and 2 on its faces. The table is given to show the possible sums achieved.

Die-1

+	1	2
1	2	3
2	3	4

Die-2

Sum	2	3	4
No. of ways	1	2	1

Also we can represent this information with use of polynomials. Each term of the polynomial represents the number on the side of the die and how many sides have that number.

We are very well familiar with six sided dice. Consider an ordinary pair of dice whose faces are labelled 1 through 6.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Ordinary Dice

+	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

Sicherman Dice

Sum	2	3	4	5	6	7	8	9	10	11	12
No. of ways.	1	2	3	4	5	6	5	4	3	2	1

How we may obtain a sum of 6 with an ordinary pair of dice. There are 5 possibilities for the two faces: (1,5), (2,4), (3,3), (4,2), (5,1). Secondly, we consider the product of the two polynomials created by using the ordinary dice labels as exponents:

Observe that we pick up the term x^6 in this product in precisely the following ways

$$(x^6 + x^5 + x^4 + x^3x^2 + x)(x^6 + x^5 + x^4 + x^3x^2 + x)$$

$$x.x^5,\ x^2.x^4,\ x^3.x^3,\ x^4.x^2,\ x^5.x$$

Notice the correspondence between pairs of labels whose sums are 6 and pairs of terms whose products are x^6 . This correspondence is one-to-one, meaning that for every pair there is one and only one product whose exponents match it. Moreover, this correspondence is valid for all sums and all dice that yield the desired probabilities.

We will let $a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6$ and $b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6$ be any two lists of positive integer labels for a pair of cubes with the property that the probability of rolling any particular sum with these dice is the same as the probability of rolling that sum with ordinary dice labelled 1 through 6. Using our observation about products of polynomials, this means that:

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)$$

$$=$$

$$+x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})\ (x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}) \ldots\ldots\ldots\ldots\ldots 1$$

Now all we have to do is solve this equation for the a's and b's. Here is where unique factorization in Z[X] comes in. The polynomial $(x^6 + x^5 + x^4 + x^3 + x^2 + x)$ factors uniquely into irreducible as

$$x(x + 1)(x^2 + x + 1)(\ x^2 - x + 1)$$

Consequently, the left-hand side of equation 1 has the irreducible factorization

$$x^2(x + 1)^2(x^2 + x + 1)^2(x^2 - x + 1)^2$$

Let $p(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$. Then by the unique factorization in Z[x], the only possible factors of $P(x)$ are

$x,\ (x+1),\ (x^2+x+1)\ and\ (\ x^2-x+1)$

This means that $P(x)$ has the form

$$x^r(x+1)^s(x^2+x+1)^t(x^2-x+1)^u$$

where $0 \leq$ r, s, t, u ≤ 2.

We can make further restrictions.

$p(x) = 1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6}) = 6$.

On the other hand $P(1) = 1^r.2^s.3^t.1^u$ and clearly we have s = 1 and t = 1

Now consider

$p(0) = 0^{a_1} + 0^{a_2} + 0^{a_3} + 0^{a_4} + 0^{a_5} + 0^{a_6} = 0$. on the other hand

$p(0) = 0^r.1^s.1^t.1^u$. If r = 0, then $0^0 = 1$ and hence $p(0) = 1$, a contradiction, therefore, r > 0. Suppose r = 2. Then the smallest possible sum one could roll with the corresponding labels for dice would be 3. Note it that we would have as x^2 in $P(x)$ as the smallest exponent, and as x in

$Q(x) = (x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6})$ as the smallest possible exponent,

Since $Q(x)$ must have positive powers of a $'s$. Thus the smallest product of $P(x)Q(x)$ is x^3 ; i.e., the

x^2 in $P(x)\ times\ x$ in $Q(x)$. But 2 is the smallest power, using L.H.S. of equation 1. Therefore, $r \neq 2$. Thus, we have reduced our possibilities to r = 1, s = 1, t = 1, and u = 0, 1, 2. We consider each case.

$when\ u = 0, P(x) = x^1(x+1)^1(x^2+x+1)^1 = x^4 + x^3 + x^3 + x^2 + x^2 + x$

so the die labels are 4, 3, 3, 2, 2, 1, a Weird die. Moreover, by equation 1 and unique factorization,

$Q(x) = x^1(x+1)^1(x^2+x+1)^1(x^2-x+1)^2 = x^8 + x^6 + x^5 + x^4 + x^3 + x$. Thus, the die labels corresponding to Q(x) are 8, 6, 5, 4, 3, 1. This is other Weird die.

When u = 1, $P(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x$, so the die labels are 6, 5, 4, 3, 2, 1 – an ordinary die. Since the factorization of this polynomial is

$x(x+1)(x^2+x+1)(\ x^2-x+1)$, using equation 1, we determine that the other die is also an ordinary one.

When u = 2, P(x) = $x^8 + x^6 + x^5 + x^4 + x^3 + x$, so the die labels are 8, 6, 5, 4, 3, 1, the other weird die. This is essentially the same case as u = 0; i.e. the other die paired with this one is the u = 0 die.

This shows that the weird dice we considered, called Sicherman dice, do give the same probabilities as ordinary dice and that they are the only other pair of dice that have this property.

Now we will try the use of the polynomial expressions to represent a four-sided die and our question is, can there be other pairs of four-sided dice that have the same sum distribution as a pair of standard four-sided dice?

Four-sided dice

Let $p(x) = x + x^2 + x^3 + x^4$ represent one standard four-sided die. To compute the distribution of sums for two standard four-sided dice, we want to compute $p(x)^2$.

Let $f(x) = p(x)^2$. Then

$$f(x) = (x + x^2 + x^3 + x^4)((x + x^2 + x^3 + x^4)$$
$$= (x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4})$$

……………..2

Now all we have to do is solve this equation for the a's and b's. Here is where unique factorization in Z[X] comes in. The polynomial $(x^4 + x^3 + x^2 + x)$ factors uniquely into irreducible as

$$x(x + 1)(x^2 + 1)$$

Consequently, the left-hand side of equation 2 has the irreducible factorization

$$x^2(x + 1)^2(x^2 + 1)^2$$

Let $p(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4}$. Then by the unique factorization in Z[x], the only possible factors of P(x) are

$x,\ (x + 1)\ and\ (x^2 + 1)$

This means that $P(x)$ has the form

$$x^r(x + 1)^s(x^2 + 1)^t$$

where $0 \leq r, s, t \leq 2$.

We can make further restrictions.

$p(x) = 1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} = 4$.

On the other hand $P(1) = 1^r.2^s.3^t$ and clearly we have s = 1 and t = 1

Now consider

$p(0) = 0^{a_1} + 0^{a_2} + 0^{a_3} + 0^{a_4} = 0$.

On the other hand

$p(0) = 0^r.1^s.1^t$. If r = 0, then $0^0 = 1$ and hence $p(0) = 1$, a contradiction, therefore, r > 0. Suppose r = 2. Then the smallest possible sum one could roll with the corresponding labels for dice would be 3. Note it that we would have as x^2 in P(x) as the smallest exponent, and as x in

$Q(x) = (x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4})$ as the smallest possible exponent.

Since $Q(x)$ must have positive powers of a $'s$. Thus the smallest product of $P(x)Q(x)$ is x^3 ; i.e., the

x^2 in $P(x)$ $times\ x$ in $Q(x)$. But 2 is the smallest power, using L.H.S. of equation 2. Therefore, $r \neq 2$. Thus, we have reduced our possibilities to r = 1, s = 2, t = 0, and r = 1, s = 0, t = 2. We consider each case. From first case we have

$P(x) = x^1(x+1)^2(x^2+1)^0 = x^3 + 2x^2 + x$

so the die labels are 3, 2, 2, 1, a Weird die. Moreover, by equation 2 and unique factorization,

when r =1, s = 0 and t =2 we have

$Q(x) = x^1(x+1)^0(x^2+1)^2 = x^5 + 2x^3 + x$. Thus, the die labels corresponding to Q(x) are 5, 3, 3, 1. This is other Weird die.

When r = 1, s =1 and t= 1 we get $P(x) = x^4 + x^3 + x^2 + x$, so the die labels are 4, 3, 2, 1 – an ordinary die. Since the factorization of this polynomial is $x(x+1)(x^2+1)$, using equation 2, we determine that the other die is also an ordinary one.

This shows that the weird dice we considered, called Sicherman dice, do give the same probabilities as ordinary dice and that they are the only other pair of dice that have this property.

This distribution shows one way to obtain a sum of 2, two ways to obtain a sum of 3, three ways to obtain a sum of 4, four ways to obtain a sum of 5, three ways to obtain a sum of 6, two ways to obtain a sum of 7 and one way to obtain a sum of 8.

The following tabular form shows this result which we have obtained above.

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

Sum	2	3	4	5	6	7	8
No. of ways	1	2	3	4	3	2	1

Here we see that the polynomial expression for f(x) and the table activity obtain the same results for each possible sum.

This arrangement of the factors is the creation of the Sicherman dice, which has a die with the numbers 1-2-2-3 and another die with the numbers 1-3-3-5. Thus there can be another way to show how the distribution of the sums can be the same as a standard pair of dice. The following chart shows Sicherman's dice sums.

Die 1

+	1	2	2	3
1	2	3	3	4
3	4	5	5	6
3	4	5	5	6
5	6	7	7	8

Sum	2	3	4	5	6	7	8
No. of ways	1	2	3	4	3	2	1

From these two systems we can obtain the same probability i.e. probability of obtaining the sum of 2 when two dice are rolled is 1/16, obtaining 3 is 2/16, 4 is 3/16, 5 is 4/16, 6 is 3/16, 7 is 2/16 and lastly probability of obtaining the sum of 8 when two dice are rolled is 1/16.

Conclusion:

The subject of Sicherman dice is both interesting in its direct connection to dice and games, but in its application to abstract algebra as well. The issue of the Sicherman labelling, their relationship to unique factorization domains, and the generalization of dice labelling into broader problems creates a great amount of depth to the subject. By slightly changing the dice problem, there are never-ending variations to explore and results to find.

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