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Study of Some Approaches for Solving Problems- External Direct Product

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Abstract:

In this paper we review briefly some of the basic results and techniques in the external direct product. Here some examples are solved to demonstrate the potential use of proposed results and techniques and we obtained solutions.

1 INTRODUCTION

Here some problems are solved by using some definitions and theorems on external direct product.

Definition

Let $G_1, G_2, G_3 \ldots\ldots G_n$ be a finite collection of groups. The external direct product of $G_1, G_2, G_3, \ldots\ldots G_n$, written as $G_1 \oplus G_2 \oplus G_3 \oplus \ldots\ldots \oplus G_n$ is the set of all n-tuples for which the i^{th} component is an element of G_i and the operation is component wise.

In symbols,

$G_1 \oplus G_2 \oplus G_3 \oplus \ldots\ldots \oplus G_n = \{(g_1, g_2, g_3 \ldots\ldots g_n) | g_i \in G_i\}$

Where $(g_1, g_2, g_3, \ldots\ldots g_n)(g_1', g_2', g_3', \ldots\ldots g_n')$ is defined to be

$$(g_1 g_1',\ g_2 g_2',\ g_3 g_3', \ldots\ldots g_n g_n')$$

Example- $Z_2 \oplus Z_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$

Theorem- The order of an element in a direct product of finite groups is the least common multiple of the orders of the components of elements.

$i.e. \left|(g_1, g_2, g_3, \ldots\ldots g_n)\right| = l.c.m.\{|g_1|, |g_2|, |g_3| \ldots\ldots |g_n|\}$

- To find, which order of subgroups does $Z_4 \oplus Z_2$ have?

For this consider,

Positive divisors of Z_4 are 1, 2, 4

Positive divisors of Z_2 are 1, 2

From this we get the set

$\{(1, 1), (1,2), (2.1), (2,2), (4,1), (4,2)\}$

Now $l.c.m(1,1)$= 1 , $l.c.m(1,2)$= 2. $l.c.m.(2,1)$= 2, $l.c.m.(2,2)$= 2 $l.c.m.(4,1)$=4, $l.c.m.(4,2)$=4.

we have repeated l.c.m.as 1, 2, and 4

So $Z_4 \oplus Z_2$ have subgroups of order 1, 2, and 4.

Example -1 How many subgroups of order 4 does $Z_4 \oplus Z_2$ have?

Number of subgroups of order 4

$$= \frac{Number\ of\ elements\ of\ order\ d}{\emptyset(d)}$$

$$= \frac{\emptyset(4)\emptyset(1) + \emptyset(4)\emptyset(2)}{\emptyset(4)}$$

$$= \frac{2 * 1 + 2 * 1}{2}$$

= 2

There are 2 subgroups of order 4 of $Z_4 \oplus Z_2$.

Example – 2 find all subgroups of order 3 in $Z_9 \oplus Z_3$.

Positive divisors of

9 - 1, 3, 9

3- 1, 3

So we have

{(1,1), (1, 3), (3,1), (3,3), (9, 1), (9,3) }

$l.c.m.$(1, 1) = 1

$l.c.m.$(1, 3)= $l.c.m.$(3, 1) =$l.c.m.$ (3, 3) =3

$l.c.m.$(9, 1)= $l.c.m.$(9, 3) = 9

∴ $Z_9 \oplus Z_3$ has subgroups of order 1, 3 and 9.

∴Number of subgroups of order 3

$$= \frac{Number\ of\ elements\ of\ order\ 3}{\emptyset(3)}$$

$$\frac{\emptyset(1)\emptyset(3) + \emptyset(3)\emptyset(1) + \emptyset(3)\emptyset(3)}{\emptyset(3)}$$

$$= \frac{1 * 2 + 2 * 1 + 2 * 2}{2}$$

$$= \frac{8}{2}$$

= 4

Thus there are four subgroups of order three. Each subgroup of order three contains one nonidentity order three element. Thus four subgroups are generated by the four nonidentity order three elements in $Z_9 \oplus Z_3$.

These four subgroups are

$$\langle(0,1)\rangle, \langle(3,0)\rangle, \langle(3,1)\rangle, \langle(3,2)\rangle$$

Example- 3 Find all subgroups of order 4 in $Z_4 \oplus Z_4$.

Here positive divisors of 4 are,

1, 2 and 4 from this we get the set of order pairs as

$$\{(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (4,1), (4,2), (4,4)\}$$

$l.c.m.$(1, 1) = 1

$l.c.m.$(1, 2) = (2, 1) = (2, 2) = 2

$l.c.m$(1, 4) = $l.c.m.$(2, 4) = $l.c.m.$(4, 1) = $l.c.m.$ (4, 2) = (4, 4) =4

∴ $Z_4 \oplus Z_4$ have subgroups of order 1, 2 and 4.

Now we will find all subgroups of order 4.

∴ Number of subgroups of order 4

$$= \frac{Number\ of\ elements\ of\ order\ 4}{\emptyset(4)}$$

$$\frac{\emptyset(1)\emptyset(4) + \emptyset(2)\emptyset(4) + \emptyset(4)\emptyset(1) + \emptyset(4)\emptyset(2) + \emptyset(4)\emptyset(4)}{\emptyset(4)}$$

$$= \frac{1 * 2 + 1 * 2 + 2 * 1 + 2 * 1 + 2 * 2}{2}$$

$$= \frac{12}{2}$$

= 6

Thus there total 6 subgroups of order 4. Each subgroup of order four contains one nonidentity order four element. Thus six subgroups are generated by the six nonidentity order four elements in $Z_4 \oplus Z_4$.

These six subgroups are

$$\langle(0,1), (1,0), (1,1), (1,2), (1,3), (2,1)\rangle$$

Example – 4 what is the largest order of any element in $Z_{30} \oplus Z_{20}$.

Solution – we know that the order of an element (a, b) ∈ $Z_{30} \oplus Z_{20}$

Is $l.c.m.$(|a|, |b|). For any (a, b) ∈ $Z_{30} \oplus Z_{20}$,

|a| divides 30 and |b| divides 20. We have the possibilities

Order (a) = 1, 2, 3, 5, 6, 10, 15, 30

Order (b) = 1, 2, 4, 5, 10, 20

The largest least common multiple among the possibilities is that of 30 and 20 which is 60.

Thus the largest order of any element in $Z_{30} \oplus Z_{20}$ is 60.

Example- 5 How many elements of order 2 are in?

$Z_{2000000} \oplus Z_{4000000}$.

Solution- $Z_2 \oplus Z_4$ and

$Z_{2000000} \oplus Z_{4000000}$

Both have same number of elements of order 2.

$Z_2 \oplus Z_4$ has total three elements of order 2. So that

$Z_{2000000} \oplus Z_{4000000}$ has three elements of order 2.

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